The Greenhouse Effect and Carbon Accumulation Dynamics: A General Equilibrium Simulation

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Introduction

The argument on the global warming through greenhouse effect gained momentum in the 1980s. Brown [1987] warned the decline of food production due to the greenhouse effect. One of the main concerns of The Earth Summit in Rio de Janeiro in 1992 was whether the participants could agree to adopt the environmental tax on CO₂ emission globally. The agreement was not reached. In 1995 the IPCC concluded that global warming is taking place due to human activity: greenhouse effect. The main aim of the 1997 Kyoto Protocol was to reach agreement for each country to reduce the global warming gases. Stern [2007] still warned the decline of food production due to the greenhouse effect.

It must be noted, however, that photosynthesis is required for the growth of plant. In other words, plants cannot grow without CO₂, and food production is zero when Y, the CO₂ level in the atmosphere, is zero how much inputs such as labor and energy are used. When Y increases from zero, the food production rises with the same inputs. Only when Y exceeds a critical level, \( \bar{Y} \), Y causes greenhouse effect, and food production decreases with the same inputs as Y (≥ \( \bar{Y} \)) increases. In this paper, this mechanism of CO₂ is explicitly introduced into the production function of food. In other words, the increase of Y provides external economy towards the food production when Y < \( \bar{Y} \), while it causes external diseconomy when Y ≥ \( \bar{Y} \).

In this paper, a dynamic process on Y is constructed under the general equilibrium framework, and the property of the stationary points of the dynamic process is examined. There are two production sectors: agriculture sector and energy sector; and one (aggregate) household in this general equilibrium model. Agriculture sector produces food utilizing labor and energy with Y as external factor, while the energy industry requires only labor to produce energy. Household consumes food and energy, supplying labor. At each time, production and consumption are made with Y given. This economic model must be constructed in the dynamic system, since the production and consumption of energy raises Y at each time, while there are two mechanisms which reduces Y at each time. One is the photosynthetic function of the trees and farm outputs and so on, and the other is a function of the sea as the greatest repository of carbon. Thus, Y is subject to countervailing factors as time elapses: one is the enhancing factor exhibited by the combustion of fossil fuels through the human activity and the other are reducing factors just mentioned. In this way, this paper develops a primitive economic model to explore the variation of CO₂ through human activity, and examine the property of the stationary points of the dynamic process. A Pigouian tax on energy is also examined.
1. SHORT-RUN GENERAL EQUILIBRIUM MODEL (SGE)

A primitive general equilibrium (GE) model is constructed, for the purpose of examining the greenhouse effect. Suppose that there are two firms. The first firm is a farm which produces wheat; \( Z_f \). Whereas wheat is produced by labor: \( L_1 \), and energy: \( H_{f1} \), the output is affected by CO2: \( Y \), in the atmosphere. Thus, this farm has the production function:

\[
Z_f = g_1[L_1, H_{f1}, Y] \quad \text{where } g_{11} > 0, \text{ and } g_{12} > 0
\]

(1)

and \( g_j \) is the partial derivative of \( g_i \) with respect to the \( j \)th variable. The second firm is the energy industry which produces energy: \( H_{f2} \), using only labor: \( L_2 \). It has the production function

\[
H_{f2} = g_2[L_2] \quad \text{where } g_{21} > 0
\]

(2)

There is only one (representative) household, which consumes wheat: \( Z_h \), and energy: \( H_h \). Household behavior is stipulated by the optimal problem:

\[
\text{max } u[Z_h, H_h]
\]

s.t. \( p Z_h + p_h H_h = w N + \pi_1 + \pi_2 \)

(3-1)

where \( u[Z_h, H_h] \) is the utility function, \( p \) is the wheat price, \( p_h \) is the energy price, \( w \) is the wage rate, \( N \) is the initial leisure hours (population), and \( \pi_i \) is the profit from the \( i \)th firm (\( i=1, 2 \)). For the sake of simplification, in this model, leisure consumption is excluded from the utility function.

Given \( Y \), the short-run General Equilibrium (SGE) is obtained, which satisfies

\[
H_{f1}^d + H_h^d = H_{f2}^s, \quad \text{energy market) (4-1)}
\]

\[
Z_h^d = Z_f^s, \quad \text{wheat market, and (4-2)}
\]

\[
L_1^d + L_2^d = N \quad \text{ (labor market) (4-3)}
\]

where superscript \( d(s) \) implies "demand" ("supply").

Suppose that production and utility functions are stipulated by the following.

\[
g_1[L_1, H_{f1}, Y] = L_1^{\alpha_1} H_{f1}^{\alpha_2} A[Y]^{\alpha_3}, \quad \alpha_1 + \alpha_2 + \alpha_3 \leq 1
\]

(5-1)

\[
g_2[L_2] = L_2
\]

(6)

\[
u[Z_h, H_h] = Z_h \gamma H_h^{1-\gamma}, \quad 0 < \gamma < 1
\]

(7)

The function, \( A[Y] \), Externality Function, is assumed to be of the one-peaked mountain shape with the following conditions.

\[
A[0] = 0, \quad A[\infty] = 0
\]

(5-2)

In the simulation of this section, it is assumed that \( A[Y] = \sin[1/(Y1000+\pi)] \).
\[ a[x_] := \sin[1/(x/1000 + 1/Pi)]; \\
Plot[a[x], \{x, 0, 3000\}, PlotRange -> All, AxesLabel -> \{"Y", "A[Y]"\}]

It is assumed that CO\(_2\) is required for the photosynthesis of wheat, while too much of it reduces the growth of wheat due to the greenhouse effect.

By (6), \(p_2 = w\) must hold at SGE, and the energy is provided by the amount demanded by the first firm and the household.

In this formulation, the relative price of energy, \(p_{H}/p\), is determined at SGE. From \(Z^d = Z_j\) in (4-2), \(p_{H}/p\) is computed from the Mathematica programming as in what follows.

\[ \text{Clear}[a]; u = z^r \ast h^\gamma (1 - r); \]
\[ \text{sol1} = \text{Solve}[[D[u, z] / D[u, h] = p / ph, p \ast z + ph \ast h = m0], \{z, h\}][[1]]; \]
\[ g1 = l1^a1 \ast h^a2 \ast a[y]^a3; \]
\[ pi1 = p \ast g1 - w \ast l1 - ph \ast h; \text{sol2} = \text{Solve}[[D[pi1, l1] = 0, D[pi1, h] = 0], \{l1, h\}]; \]
\[ l10 = \text{Simplify}[\text{Factor}[l1 /. \text{sol2}[1][]])); \]
\[ h0 = \text{Simplify}[\text{Factor}[h /. \text{sol2}[1][]])); \]
\[ z0 = \text{Simplify}[\text{PowerExpand}[g1 /. \{l1 -> l10, h -> h0\}]]; \]
\[ pi10 = \text{Simplify}[\text{PowerExpand}[pi1 /. \{l1 -> l10, h -> h0\}]]; \]
\[ \text{sol11} = \text{Simplify}[\text{sol1} /. \text{m0} \rightarrow w \ast n0 \ast pi10 /. \text{w} \rightarrow \text{ph}]; \]
\[ \text{check1} = \]
\[ \text{PowerExpand}[\text{Simplify}[\text{Expand}[[z0 / . (w \rightarrow \text{ph})] = (z / . \text{sol11} / . (w \rightarrow \text{ph}))]]]; \]
\[ \text{sol3} = \text{PowerExpand}[\text{Simplify}[\text{Solve}[\text{check1}, \text{p}][[1]])]

\[ \{p \rightarrow a1^{-\pi1} \ast a2^{-\pi2} \ast n0^{-\pi3} \ast ph \ast r^{-\pi4} \}
\[ \left(a1^{-\pi1} \ast a2^{-\pi2} \ast r + a1^{-\pi1} \ast a2^{-\pi2} \ast \left(-a2^{-\pi3} \ast (1 + r) + a2^{-\pi3} \ast r\right)\right)^{-\pi4} \ast a[y]^{-\pi5}\]

Actually, the SGE relative commodity price, RCP, is given by the following.

\[ p_{H}/p = ((1 - \gamma(1 - \alpha_1 - \alpha_2))/(\gamma N)^{\pi2} - a1^{\alpha1} \ast a2^{\alpha2} \ast A[Y]^{\alpha1}) \quad (8)\]

To show this, on the one hand, when parameters are randomly specified by

\[ \alpha_1 = 1/2, \alpha_2 = 1/4, \alpha_3 = 1/5 \quad (9-1)\]

RCP, defined by (8) is computed as the following.
Out[14]= \[4\] rcp = ((1 - r (1 - a1 - a2)) / (r * n0)) ^ (1 - a1 - a2) * a1^a1 * a2^a2 * a[y]^a3;
rcp / . {a1 -> 1 / 4, a2 -> 1 / 4, a3 -> 1 / 4, r -> 1 / 2}

\[
\frac{\sqrt{3}}{2} \frac{1}{n0} a[y]^{1/4}
\]

Out[14]=

On the other hand, the solution, derived as sol3 from the Mathematica programming in this section, produces equilibrium price as the following when the parameters are specified by (9-1). Thus, sol3 gives rise to the SGE relative commodity price, as defined by (8).

In[15]= 
Simplify[sol3 / . {a1 -> 1 / 4, a2 -> 1 / 4, a3 -> 1 / 4, r -> 1 / 2}]

Out[15]=

It is confirmed that the equilibrium is guaranteed in the labor market. Indeed, the labor demand; \( L_1^d + L_2^d \), in (4-3), when the wheat price is given by (8) (or sol3), is computed as in what follows. It is equal to \( N \).

In[16]= 
hh = h / . sol11;
Simplify[Simplify[PowerExpand[(h0 + hh + 110) / . sol3]] / . (w -> ph)]

Out[17]=

The SGE labor input for wheat, \( L_1^E \), is given by the following.

\( L_1^E = \alpha_1 \gamma N / (1 - \gamma (1 - a_1 - a_2)) \)

In order to show this, on the one hand, when parameters are randomly specified by (9-1), \( L_1^E \), defined by (10) is computed as the following.

In[18]= 
l1E = a1 * r * n0 / (1 - r (1 - a1 - a2)); l1E / . {a1 -> 1 / 4, a2 -> 1 / 4, a3 -> 1 / 4, r -> 1 / 2}

Out[18]=

On the other hand, \( L_1^E \), computed in the Mathematica programming, is derived as the following.

In[19]= 
sol4 = Simplify[
Factor[Simplify[PowerExpand[Factor[l1 /. sol2 /. w -> ph /. sol3[[1]]]]] [[1]]]]

Out[19]=

When parameters are randomly specified by (9-1), \( L_1^E \), derived in the Mathematica programming, becomes as in the following. Thus (10) is confirmed.

In[20]= 
Simplify[sol4 / . {a1 -> 1 / 4, a2 -> 1 / 4, a3 -> 1 / 4, r -> 1 / 2}]

Out[20]=

The SGE energy input for wheat, \( H_1^E \), is given by the following.
\[ H_1^E = a_2 \gamma N / \{1 - \gamma (1 - a_1 - a_2)\} \]  

(11)

In order to show this, on the one hand, when parameters are randomly specified by (9), \( H_1^E \), defined by (11), is computed as the following.

\[
hf1E = a_2 * r * n0 / (1 - r (1 - a1 - a2)); \\
hf1E /. \{a1 \to 1/4, a2 \to 1/4, a3 \to 1/4, r \to 1/2\}
\]

Out[21]=

\[
\text{Out}[21]= \frac{n0}{6}
\]

On the other hand, \( H_1^E \), computed through the Mathematica programming, is derived as the following.

\[
\text{In}[22]:= \text{sol5} = \text{Simplify[}
\text{Factor[}
\text{Simplify[}
\text{PowerExpand[}
\text{Factor[}
\text{h / . sol2 / . w \rightarrow ph / . sol3[[1]]]]]]]]]]
\]

Out[22]=

\[
\text{Out}[22]= \left(\frac{a1 - 1 - a1 + a2}{a1 - a1 - a2} a2^{\frac{1}{2}} - n0 r + a1^{\frac{1}{2}} a2^{\frac{1}{2}} r + a2^{\frac{1}{2}} (1 + r) + a2^{\frac{1}{2}} r\right)
\]

When parameters are randomly specified by (9), \( H_1^E \), derived in the Mathematica programming, becomes as what follows. Thus (11) is confirmed.

\[
\text{In}[23]:= \text{Simplify[sol5 / . \{a1 \to 1/4, a2 \to 1/4, a3 \to 1/4, r \to 1/2\}}
\]

Out[23]=

\[
\text{Out}[23]= \frac{n0}{6}
\]

The SGE energy consumption of household, \( H_h^E \), is given by the following.

\[ H_h^E = (1 - \gamma) N / (1 - \gamma (1 - a_1 - a_2)) \]  

(12)

In order to show this, on the one hand, when parameters are randomly specified by (12), \( H_h^E \), defined by (12) is computed as the following.

\[
\text{In}[24]:= \text{hhE} = (1 - r) * n0 / (1 - r (1 - a1 - a2)); \text{hhE} / . \{a1 \to 1/4, a2 \to 1/4, a3 \to 1/4, r \to 1/2\}
\]

Out[24]=

\[
\text{Out}[24]= \frac{2 \ n0}{3}
\]

\[
\text{In}[25]:= \text{sol6} = \text{Simplify[}
\text{PowerExpand[}
\text{h / . sol11 / . sol3]}
\]

Out[25]=

\[
\text{Out}[25]= \left(\frac{a1^{\frac{1}{2}} - 1 - a1 + a2}{a1 ^{\frac{1}{2}} - a1 - a2} a2^{\frac{1}{2}} - n0 (-1 + r) + a1^{\frac{1}{2}} a2^{\frac{1}{2}} r + a2^{\frac{1}{2}} (-1 + r) + a2^{\frac{1}{2}} r\right)
\]

\[
\text{In}[26]:= \text{Simplify[sol6 / . \{a1 \to 1/4, a2 \to 1/4, a3 \to 1/4, r \to 1/2\}]
\]

Out[26]=

\[
\text{Out}[26]= \frac{2 \ n0}{3}
\]

Thus, energy consumption in this economy, \( H^E \), is the sum of \( H_1^E \) and \( H_h^E \).

\[ H^E = H_1^E + H_h^E = (1 - \gamma + a_2 \gamma) N / (1 - \gamma (1 - a_1 - a_2)) \]  

(13)

For the later use in the comparison with Pigouvian taxing case, the GE utility level, \( n_{00} \), is computed.

\[
\text{In}[27]:= \text{u00} = \text{Simplify[}
\text{PowerExpand[u / . sol11 / . sol3 / . \{a1 \to 1/4, a2 \to 1/4, a3 \to 1/4, r \to 1/2\}]}
\]

Out[27]=

\[
\text{Out}[27]= \frac{2^{1/4} n0^{3/4} a[y]^{1/8}}{3^{3/4}}
\]
Also for the later use, the GE wheat output level, \( g_{1E} \), is computed.

\[
g_{1E} = a_1 \gamma (1 - \gamma ) \left( 1 - \gamma (1 - a_1 - a_2)^{\alpha_1 + \alpha_2} \right) A^{\alpha_1 + \alpha_2}, A(Y)^{\alpha_3}
\]

```
In[29]:= Clear[a];
g1E = Simplify[PowerExpand[g1 /. \{h \rightarrow hf1E, 11 \rightarrow 11E\}]]
```

```
Out[29]= a1^2 a2^2 n0^2 a2 (1 + \(-1 + a1 + a2\) r)^{-a1-a2} a[y]^3
```

2. DYNAMIC SYSTEM, INSTABILITY AND PIGOUVIAN TAX: LONG-RUN GENERAL EQUILIBRIUM DYNAMICS (LGED)

The analysis in the previous section is called the short-run general equilibrium model, SGE, since CO2: \( Y \), in the atmosphere, is fixed by the assumption. In fact, CO2 in the atmosphere increases through the use of energy in the household's direct consumption and farm's use of energy in the wheat production, while it decreases thanks to the absorption by the working of sea and the photosynthetic function of wheat. The variation of CO2 in the atmosphere, in turn, causes the variation wheat output. Thus, the economic analysis of greenhouse effect must be constructed in terms of dynamic system. This dynamic system is called the long-run general equilibrium dynamics (LGED). In this section, an extension of this sort is attempted.

Formally, as energy is produced, \( Y \): CO2 in the atmosphere, increases by the amount of \( F_1[H] \), while \( Y \) decreases, first, by the amount of \( F_2[Y] \), thanks to the activity of the sea and, second, \( F_3[g_1] \), thanks to the photosynthetic function of wheat.

Thus, we have a dynamic system

\[
dY[t]dt = F_1[H[t]] - F_2[Y[t]] - F_3[g_1[t]] \quad (14)
\]

where \( t \) stands for time.

### 2.1 Stability Analysis and Pigouvian Tax

First of all, we examine if (14) is stable. In order to do so, suppose that

\[
F_1[H[t]] = \mu_1 H[t], \mu_1 > 0, \text{constant,} \quad (15)
\]

\[
F_2[Y[t]] = \mu_2 Y[t], \mu_2 > 0, \text{constant,} \quad (16-1)
\]

\[
F_3[g_1[t]] = \mu_3 g_1[t], \mu_3 > 0, \text{constant,} \quad (16-2)
\]

Under (14), (15), (16-1), and (16-2) the result in the previous section gives rise to

\[
dY[t]dt = \Phi[Y[t]] = \mu_1 (1 - \gamma \gamma ) N/(1 - \gamma (1 - a_1 - a_2)) - \mu_2 Y[t] - \\
\mu_3 [a_1 (1 - \gamma )^{\alpha_1} \gamma^{\alpha_1} (1 - \gamma (1 - a_1 - a_2))^{\alpha_1 + \alpha_2}] N^{\alpha_1 + \alpha_2} A(Y)^{\alpha_3} \quad (17)
\]

which is globally stable, since by assumption (5-2), \( \Phi[0] > 0 \) and \( \Phi[\infty] < 0 \) hold. Note that \( \Phi[Y[t]] \) is not necessarily a monotone decreasing function.

Indeed, on the one hand, when

\[
a_1 = a_2 = a_3 = 1/4, \gamma = 1/2, \mu_1 = 1/1000, \mu_2 = 1/1000, \mu_3 = 10, N = 100000000 \quad (9-2)
\]
there are 3 stationary points, \{18.7197, 5427.61, 6.91778 \times 10^6\}, the second element of which is locally unstable.

\(a_1=a_2=a_3=1/4, \gamma=1/2, \mu_1=1, \mu_2=1, \mu_3=10, N=1000000\) (9-3)

On the other hand, when

\[\alpha_1=\alpha_2=\alpha_3=1/4, \gamma=1/2, \mu_1=1, \mu_2=1, \mu_3=10, N=1000000\]

there is only one stationary point, \{8.33142 \times 10^6\}, which is locally stable.

It is examined if the Pigouvian tax could reduce stable stationary points of CO2. The primitive general equilibrium model in the previous section is modified as follows. As above, there are two firms. The first firm is a farm which produces wheat, with the production function specified by (1). The second firm is the energy industry which produces energy, with the production function specified by (2). The modification consists in the imposition of Pigouvian tax on the consumption of energy, while the tax revenue is distributed to the households.

There is only one (representative) household, which consumes wheat: \(Z_h\), and energy: \(H_h\). Household's behavior is stipulated by the optimal problem:

\[
\text{max } u[Z_h, H_h] \\
\text{s.t. } p_z Z_h + p_h (1+\tau) H_h = w N + \pi_1 + \pi_2 + T = M \tag{3-2}
\]

where \(T\) is tax revenue and it is equal to \(\tau p_h H_h\). Given CO2: \(Y\), the short-run General Equilibrium (GE) is obtained, which satisfies (4-1), (4-2), and (4-3).

Suppose that production and utility functions are stipulated by (5-1), (6), and (7). As in the previous section, by (6), \(p_H=w\) must hold at GE, and the energy is provided by the amount demanded by the first firm; farm, and the household. In this formulation, the relative price of energy, \(p_H/p\), is determined at GE. From \(Z_h^*=Z_h^\text{in} (4-2)\), \(p_H/p\) is computed from the Mathematica programming as in what follows. Given income, \(M\), household's demand functions for wheat and energy are computed as follows.
\[
\begin{align*}
\text{In[36]:=} & \quad u = z^r h^{(1-r)}; \\
& \quad \text{soll} = \\
& \quad \text{Solve}[\{D[u, z] / D[u, h] := p / (ph \times (1 + t)), p \times z + ph \times (1 + t) \times h = m0\}, \{z, h\}][[1]] \\
\text{Out[37]} = & \quad \left\{ \begin{array}{l}
\quad z \rightarrow \frac{m0 \times r}{p}, \quad h \rightarrow \frac{m0 \times (1 + r)}{ph \times (1 + t)}
\end{array} \right\}
\end{align*}
\]

From the profit maximization of the farm, its demand for labor, \(L_1^{d}\), demand for energy, \(H_1^{d}\), supply of wheat, \(Z_f^s\), and its profit, \(\pi_1\), are computed as in what follows.

\[
\begin{align*}
\text{In[38]:=} & \quad \text{Clear[}a, g1, pil, sol21, sol22, h0, sol23, l10, z0, pil0\}; \\
& \quad g1 = 11^a1 \times h^a2 \times a[y]^a3; \\
& \quad pil = p \times g1 - w \times l1 - ph \times (1 + t) \times h; \\
& \quad sol21 = \text{PowerExpand[Solve[D[pil, l1] \rightarrow 0, l1][[1]]]}; \\
& \quad sol22 = \\
& \quad \text{PowerExpand[Simplify[PowerExpand[(D[pil, h] / sol21)] \rightarrow 0, h][[1]]]}; \\
& \quad h0 = h / . \text{sol22}; \\
& \quad sol23 = \text{Simplify[PowerExpand[sol21 / . sol22]]}; \\
& \quad l10 = l1 / . \text{sol22}; \\
& \quad z0 = \text{Simplify[PowerExpand[g1 / . \{l1 \rightarrow l10, h \rightarrow h0\}]];} \\
& \quad pil0 = \text{Simplify[PowerExpand[pil / . \{l1 \rightarrow l10, h \rightarrow h0\}]];} \\
& \quad \{"11" \rightarrow l10, "h" \rightarrow h0, "z" \rightarrow z0, "pil" \rightarrow pil0\}
\end{align*}
\]

\[
\begin{align*}
\text{Out[46]} = & \quad \left\{ \begin{array}{l}
\quad l1 \rightarrow a1 \times a2 \times a[y]^{-a3}, \\
\quad h \rightarrow a1 \times a2 \times a[y]^{-a3}, \\
\quad z \rightarrow a1 \times a2 \times a[y]^{-a3}, \\
\quad pil \rightarrow a1 \times a2 \times a[y]^{-a3}, \\
\quad \frac{1}{1-a2} \times ph \times a[y]^{-a3}
\end{array} \right\}
\end{align*}
\]

GE relative price, \(p_H/p\), is computed by solving

\[
\begin{align*}
& w N + \pi_1 + \tau p_H H_1 + \tau p_H H_2 = M, \quad (3-2) \\
& Z_{a1}^d = Z_f^s \quad \text{(wheat market equilibrium)} \quad (4-2)
\end{align*}
\]

First of all, \(M\), which satisfies (4-2), is derived as a function of \(p_H\) and \(p\). Subsequently, substituting this function into (3-2), we derive GE relative price, \(p_H/p\), as in what follows.
In[47]:= checkA = Solve[{z0 /. {w -> ph}} == (z /. sol1), m0][[1]]; m01 = Simplify[w*n0 + pi0 + t*ph*h0 + t*ph*h /. sol1 /. w -> ph /. checkA]; checkB = Simplify[(m0 /. checkA) == m01]; sol3 = Simplify[Solve[checkB, p][[1]]]]

Out[50]= \[\begin{align*}
p & \rightarrow -a_1^{-1-a_1} a_2^{-1-a_2} n_0^{-1-a_1-a_2} p h^{-1-a_1-a_2} \\
& \quad + r^{-1-a_1-a_2} (l + t)^{-a_1} \left(a_1 \frac{1}{a_2^{-1-a_1-a_2}} a_2^{-1-a_2} \frac{1}{a_1^{-1-a_1-a_2}} + \right) \\
& \quad \left(a_1 \frac{1}{-1-a_1-a_2} a_2^{-1-a_2} \left(-1 + r \right) \left(-t \left(l + t\right)^{-1} + (l + t)^{-1} \right) + a_1 \frac{1}{a_2^{-1-a_1-a_2}} a_2^{-1-a_2} \frac{1}{a_1^{-1-a_1-a_2}} \right) + \\
& \quad a_1^{-1-a_1-a_2} \left(a_2^{-1-a_1-a_2} \frac{1}{a_1^{-1-a_1-a_2}} (-1 + r) \left(-t \left(l + t\right)^{-1} + (l + t)^{-1} \right) + a_2^{-1-a_1-a_2} \left(a_1 \frac{1}{a_2^{-1-a_1-a_2}} a_2^{-1-a_2} \frac{1}{a_1^{-1-a_1-a_2}} \right) \right)
\end{align*}\]

It is confirmed that the equilibrium is guaranteed in the labor market. Indeed, the labor demand; \(L_1^d+L_2^d\), in (4-3), when the wheat price is given by (8) (or sol3), is computed as in what follows. It is equal to \(N\).

In[51]:= hh = h /. sol1; Factor[Simplify[
Simplify[PowerExpand[\((h_0 + hh + 110)\)] /. m0 -> m01 /. sol3 /. {w -> ph}]] /. 
\{a_1 -> 1/4, a_2 -> 1/4, a_3 -> 1/4, r -> 1/2, t -> 2/10\}

Out[52]= n0

It is also confirmed that the equilibrium when is \(\tau=0\) is the same as the one derived in the previous section.

In[53]:= PowerExpand[sol3 /. t -> 0 /. 
\{a_1 -> 1/4, a_2 -> 1/4, a_3 -> 1/4, r -> 1/2, m_1 -> 1/1000, m_2 -> 1/1000, m_3 -> 10\}

Out[53]= \[\begin{align*}
p & \rightarrow \left(\frac{2}{3} \sqrt{n_0} \right) \frac{ph}{a[y]^{1/4}} 
\end{align*}\]

The GE energy consumption for producing wheat, \(H_1^E\), computed in the Mathematica programming, is derived as the following.

In[54]:= Hf1t = Simplify[h_0 /. sol3 /. w -> ph /. 
\{a_1 -> 1/4, a_2 -> 1/4, a_3 -> 1/4, r -> 1/2, m_1 -> 1/1000, m_2 -> 1/1000, m_3 -> 10\}

Out[54]= \[\frac{n_0}{6 + t}\]

The GE energy consumption by the household, \(H_2^E\), computed in the Mathematica programming, is derived as the following.

In[55]:= Hht = Simplify[hh /. {m0 -> m01} /. sol3 /. 
\{a_1 -> 1/4, a_2 -> 1/4, a_3 -> 1/4, r -> 1/2, m_1 -> 1/1000, m_2 -> 1/1000, m_3 -> 10\}

Out[55]= \[\frac{4 \ n_0}{6 + t}\]

Thus, as \(\tau\) rises, \(H_1^E+H_2^E\) declines, so that it appears that the RHS of (14) shifts downward. Indeed, it is the case, since the wheat output rises due to the GE price modification. Note that in the traditional microeconomics, when only the price of one input rises with other prices constant, all the inputs decline and because of this output also declines. However, in the
GE framework, the prices change, and labor input for the farm rises.

\[ \text{Out}[59]= \]

\[ \text{In}[62]:= \]

\[ \text{Out}[62]= \]

Thus, wheat output rises due to the tax imposition on energy at GE framework.

\[ \text{Out}[63]= \]

\[ \text{Out}[64]= \]

In this way, the RHS of (14) shifts downward, and the values of stable stationary points declines. Under (9.2) with \( \tau = 1/10 \), there are 3 stationary points, \( \{16.3287, 6225.04, 6.75012 \times 10^6\} \), the second element of which is locally unstable.

\[ \text{Out}[65]= \]

\[ \text{Out}[66]= \]

\[ \text{Out}[67]= \]

A remark is in order here. While the Pigouvian taxing policy can reduce the values of stable stationary points, does it imply that the utility level rises? In order to examine this problem, let \( u \) be the short-run GE utility level, depending on the tax rate, \( \tau \), given \( \text{CO}_2 : Y \). When (9) is assumed, it is computed as in what follows. When \( \tau = 0 \), it is confirmed that \( u_0 = u_{00} \).

\[ \text{Out}[68]= \]

\[ \text{Out}[69]= \]

\[ \text{Out}[70]= \]

\[ \text{Out}[71]= \]

\[ \text{Out}[72]= \]

It is easy to show that \( d u / d \tau < 0 \). Thus, given \( \text{CO}_2 : Y \), as the Pigouvian tax rate rises, the GE utility level declines.

\[ \text{Out}[73]= \]

\[ \text{Out}[74]= \]

\[ \text{Out}[75]= \]

On the one hand, under (9-2), when \( \tau = 0 \), the GE utility levels at the three stationary points are computed as in what follows, respectively.

\[ \text{Out}[76]= \]

\[ \text{Out}[77]= \]

\[ \text{Out}[78]= \]

Also, under (9-2), when \( \tau = 1/10 \), the GE utility levels at the three stationary points are computed as in what follows,
respectively. At the lower stationary point, the utility level declines due to the Pigouvian tax, while at the highest stationary point, the utility level rises. This result could be explained as in what follows.

In[65]:= \text{In[66]} \rightarrow 
\text{Out[66]} = 
\text{In[67]} := 
\text{In[68]} := 
\text{In[69]} := 

At the lower stationary point, if the \( CO_2 : Y \) rises, the output rises, which may contribute to the improvement of utility level. The Pigouvian tax reduces \( CO_2 : Y \), which leads to the final result of reduced utility. On the other hand, at the higher stationary point, if the \( CO_2 : Y \) rises, the output declines, which may contribute to the reduced utility level. The Pigouvian tax reduces \( CO_2 : Y \), which leads to the final result of improved utility.

Under (9.3), there is only one stationary point. When \( \tau = 0 \), the GE utility level at the unique stationary point is computed as in what follows.

In[70]:= \text{In[71]} \rightarrow 
\text{Out[71]} = 
\text{In[72]} := 
\text{In[73]} := 
\text{In[74]} := 
\text{Out[75]} = 

At the unique stationary point, if the \( CO_2 : Y \) rises, the output declines, which may contribute to the worsened utility level. The Pigouvian tax reduces \( CO_2 : Y \), which leads to the final result of improved utility.

\section{2.2 Structure of Stationary Points With Respect To \( \mu_3 \)}

In the previous subsection, given \( \mu_3 = 10 \), it was shown that two cases are possible on the stationary points. The first case is that when \( \mu_1 = \mu_2 = 1 \), there are two stable stationary points and one unstable stationary point. The second case is that when \( \mu_1 = \mu_2 = 1 \), there is only one stationary point. In this subsection, we examine how the structure of stationary points varies depending on \( \mu_3 \), while assuming that \( a_1 = a_2 = a_3 = 1/4 \), \( \gamma = 1/2 \), and \( \mu_1 = \mu_2 = 1/1000 \), \( N = 10000000 \). The following function, \text{ff1[z]}, is the right hand side of (17) where \( y = Y[t] \) and \( \mu_1 = \mu_2 = 1/1000 \), \( N = 10000000 \), arbitrarily given \( \mu_3 = z \).

\begin{verbatim}
In[76]:= \text{Clear[}A1, \text{ff1, solE0, check1]} 
In[77]:= \text{ff1[m30_,]} := 
\text{Module[\{A1, f, a\}, A1 = \{a1 \rightarrow 1/4, a2 \rightarrow 1/4, a3 \rightarrow 1/4, r \rightarrow 1/2, n0 \rightarrow 10 000 000, m1 \rightarrow 1/1000, m2 \rightarrow 1/1000, m3 \rightarrow m30\}; a[x_] := \text{Sin[}1/\text{(x/1000} + 1/\text{Pi})\}; 
\text{f = (}m1 (1 - r + a2 \times r) \times n0 / (1 - r (1 - a1 - a2)) - m2 \times y - m3 ( (a1 \times a1) \times (a2 \times a2) \times (r \times (a1 + a2)) / ((1 - r (1 - a1 - a2)) \times (a1 + a2)) \} \times (n0 \times (a1 + a2)) \times (a[y] \times a3) \}/. A1] 
\end{verbatim}

\subsection{2.2.1 \( \mu_3 = 1/10 \)}

When \( \mu_3 \) is small, say 1/10, \text{ff1[}\mu_3\text{]} consists of two extreme points: \( y_1 \) and \( y_2 \). The smaller one, \( y_1 \), is around 500, as shown by the following graph.
The greater one, \( y_2 \), is around 16000, as shown by the following graph.

There is only one stationary point, \( y^* \): \( f\hat{1}[\mu_3]=0 \); when \( y^* \) is around \( 8.3 \times 10^6 \), as shown by the following graph.

The stationary point, \( y^* \): \( f\hat{1}[\mu_3]=0 \); is easily computed by the Newton method.
\begin{verbatim}
In[73]= sol[1/10] = FindRoot[ffl[1/10] = 0, {y, 1000000}, MaxIterations \rightarrow 1000, WorkingPrecision \rightarrow 20]
Out[73]= \{y \rightarrow 8.3198159761241829714 \times 10^6\}
\end{verbatim}

\textbf{2.2.2} $\mu_3=10$

When $\mu_1$ is not small, neither large, say 10, $ffl[\mu_1]$ consists of two extreme points: $y_1$ and $y_2$ as in the previous case. The smaller one, $y_1$, is around 500, as shown by the following graph.

\begin{verbatim}
In[74]= Plot[ffl[10], {y, 0, 100000}]
\end{verbatim}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Graph showing the behavior of $ffl[10]$}
\end{figure}

The greater one, $y_2$, is around 500000, as shown by the following graph.

\begin{verbatim}
In[75]= Plot[ffl[10], {y, 100000, 8000000}]
\end{verbatim}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Graph showing the behavior of $ffl[10]$ (large values)}
\end{figure}

As shown in 2.1, we have three stationary points, $y^{s1}$, $y^{s2}$, and $y^{s3}$: $ffl[\mu_3]=0 \ (i=1,2,3)$; and they are easily computed by the Newton method. The greatest stationary point: $y^{s3}$ is computed as in what follows.

\begin{verbatim}
In[76]= sol[1/10] = FindRoot[ffl[10] = 0, {y, 1000000}, MaxIterations \rightarrow 1000, WorkingPrecision \rightarrow 20]
Out[76]= \{y \rightarrow 6.9177750007024042247 \times 10^6\}
\end{verbatim}

\textbf{2.2.3} $\mu_3=80$

When $\mu_3$ is large, say 80, $ffl[\mu_3]$ consists of two inflection points: $y_1$ and $y_2$ as in the previous case. The smaller one, $y_1$, is around 500, as shown by the following graph.
The greater one, $y_2$, is around 3000000, as shown by the following graph.

There is only one stationary point, $y^*$: $f_1[y_3]=0$; when $y^*$ is around 0.004, as shown by the following graph.

This stationary point, $y^*$: $f_1[y_3]=0$; is computed by the Newton method.
2.2.4 $\mu_3=6.45$ and 33

Note that there are two values of $\mu_3$: $\mu_3^1 < \mu_3^2$; at which extreme point coincides with stationary point. The smaller point, $\mu_3^1$, is around 6.45, with the extreme point (=stationary point) is at around $y^*=300$, as shown by the following graph.

The greater point, $\mu_3^2$, is around 33, with the extreme point (=stationary point) is at around $y^*=1.7 \times 10^6$, as shown by the following graph.

The following program produces the graph of the combination of $\mu_3$ and the stationary points corresponding to each $\mu_3$. In the figure point B corresponds to the combination of $\mu_3^2$ and the corresponding stationary points, while point C corresponds to the combination of $\mu_3^1$ and the corresponding stationary points. Note that point B is an inflection point itself. Note furthermore that on the curve BC stationary points show local instability.
It is revealed in the following that the point C itself is an inflection point. Thus, the graph, A-B-C-D is actually of the inverse S shape.

As the parameter $\mu_3 = z$ moves the corresponding stable CO$_2$ level, $Y^*$, shows discontinuous movement. On the one hand, as $\mu_3$ increases from zero, the corresponding $Y^*$ decreases, moving on the AB curve. When $\mu_3$ increases further, however, and it crosses $\mu_3^*$, the corresponding $Y^*$ decreases, moving on the CD curve. Thus, discontinuity exists in this movement.
On the other hand, as $\mu_1$ decreases from 100, the corresponding $Y^*$ increases, moving on the CD curve. When $\mu_3$ decreases further, however, and it crosses $\mu_3^1$, the corresponding $Y^*$ increases, moving on the AD curve. The discontinuity also exists in this movement. Thus we have two discontinuous points in this mapping.

3. SHORT-RUN GENERAL EQUILIBRIUM MODEL (SGE):
Different Externality Function Case

In this section, a different Externality Function is used in order to examine the robustness of the previous section. In the simulation, it is assumed that $A[Y]$ is a normal distribution with expected value of 1000 and the standard deviation of 100:

$$A[Y] = \frac{\exp \left( \frac{-100(Y-1000)^2}{20000} \right)}{100 \sqrt{2 \pi}}$$

As in the previous sections, it is assumed that CO$_2$ is required for the photosynthesis of wheat, while too much of it reduces the growth of wheat due to the greenhouse effect.

As in the previous sections, by (6), $p_H$ must hold at SGE, and the energy is provided by the amount demanded by the first firm and the household. In this formulation, the relative price of energy, $p_H/p$, is determined at SGE. From $Z_0^d=Z_0^f$ in (4-2), $p_H/p$ is computed as in (8). As in the previous sections, the SGE labor input for wheat, $L_1^E$, is given by (10). The SGE energy input for wheat, $H_1^E$, is given by (11). The SGE energy consumption of household, $H_h^E$, is given by (12). Thus, energy consumption in this economy, $H^E$, is the sum of $H_1^E$ and $H_h^E$, and (13) holds. Furthermore, the GE wheat output level, $g_{1h}$, is given by the same function as in the previous sections.

In exactly the same way, under (14), (15), (16-1), and (16-2) we have the same dynamic process on the CO$_2$ level as in the previous section.

$$dY[t]/dt=\Phi[Y[t]]=\mu_1(1-y+\alpha_2 Y)/(1-y(1-\alpha_1-\alpha_2))-\mu_2 Y[t]=-
\mu_3(\alpha_1^o(1-\gamma)^{\alpha_1^o}/(1-\gamma(1-\alpha_1-\alpha_2))^{\alpha_2^o})^{\alpha_0^o} Y^{1/\alpha_0^o}, \Phi[Y[t]] \ (17)$$

which is globally stable, since by assumption (5-2), $\Phi[0]>0$ and $\Phi[\infty]<0$ hold. Also, as in the previous sections, $\Phi[Y[t]]$ is not necessarily a monotone decreasing function.

The following function, $ff2[z]$, is the right hand side of (17) where $y=Y[t]$ and $\alpha_1=\alpha_2=\alpha_3=1/4$, $\gamma=1/2$, and $\mu_1=\mu_2=1/1000$, $N=10000000$, arbitrarily given $\mu_3=3$.

In[99]:= Clear[A1, ff1, solE0, check1]
\[ ff2[m30_] := \text{Module}"(A1, f, a),
\]
\[ A1 = (a1 \rightarrow 1/4, a2 \rightarrow 1/4, a3 \rightarrow 1/4, r \rightarrow 1/2, n0 \rightarrow 10000000, m1 \rightarrow 1/1000, m2 \rightarrow 1/1000, m3 \rightarrow m30); a[x_] := \text{PDF}[	ext{NormalDistribution}[1000, 100], x]; \]
\[ f = (m1 (1 - r + a2 * r) * n0 / (1 - r (1 - a1 - a2))) - m2 * y - m3 (((a1^a1) * (a2^a2) * (r^a1 + a2)) / ((1 - r (1 - a1 - a2)) ^ (a1 + a2))) * (n0^a1 + a2)) * (a[y]^a3) / . A1 \]

\subsection{3.1 \( \mu_3 = 1/10 \)}

When \( \mu_3 \) is small, say \( 1/10 \), \( ff2[\mu_3] \) consists of two extreme points: \( y_1 \) and \( y_2 \). The smaller one, \( y_1 \), is around 1000, as shown by the following graph.

\begin{center}
\textbf{Plot}[ff2[1/10], \{y, 0, 2000]\}]
\end{center}

![Graph showing \( ff2[1/10] \) with \( y_1 \) around 1000]

The greater one, \( y_2 \), is around 1700, as shown by the following graph.

\begin{center}
\textbf{Plot}[ff2[1/10], \{y, 1600, 1900\}]
\end{center}

![Graph showing \( ff2[1/10] \) with \( y_2 \) around 1700]

There is only one stationary point, \( y^* \): \( ff2[\mu_3] = 0 \); where \( y^* \) is around \( 8.3 \times 10^6 \), as shown by the following graph.

\begin{center}
\textbf{Plot}[ff2[1/10], \{y, 1650, 1900\}]
\end{center}

![Graph showing \( ff2[1/10] \) with a single stationary point]
The stationary point, $y^*$: \( f_2[\mu_1] = 0 \), is easily computed by the Newton method.

When \( \mu_3 \) is not small, neither large, say 30, \( f_2[\mu_3] \) consists of two extreme points: \( y_1 < y_2 \). The smaller extreme point, \( y_1 \), is around 1000, as shown by the following graph.
There are three stationary points, \( y^1 < y^2 < y^3 \): \( ff2[m_3] = 0 \) (\( i = 1, 2, 3 \)). The smaller stationary points, \( y^1 < y^2 \), are already depicted in the first graph of this sub-section. The greatest one, \( y^3 \), is around \( 8.3 \times 10^6 \), as shown by the following graph.

The stationary points, \( y^i \): \( ff2[m_3] = 0 \) (\( i = 1, 2, 3 \)); are easily computed by the Newton method.

When \( m_3 \) is large, say 100, \( ff2[m_3] \) consists of two extreme points: \( y_1 < y_2 \). The smaller extreme point, \( y_1 \), is around 1000, as shown by the following graph.
The greater extreme point, \( y_2 \), is around 2000, as shown by the following graph.

There are three stationary points, \( y^1 < y^2 < y^3 \): \( \text{ff2}[\mu_3]=0 \) \((i=1,2,3)\). The two smaller stationary points, \( y^1 < y^2 \), are already depicted in the first graph of this sub-section. The greatest one, \( y^3 \), is around \( 8.3 \times 10^6 \), as shown by the following graph.

Three stationary points, \( y^1 < y^2 < y^3 \): \( \text{ff2}[\mu_3]=0 \) \((i=1, 2, 3)\) are computed by the Newton method.
\textbf{3.4 }\mu_3= 25.6

Note that there is a value of \( \mu_3: \tilde{\mu}_3 \) at which extreme point coincides with stationary point. This value, \( \tilde{\mu}_3 \), is a unique point, around 25.6, with the extreme point (minimum point=stationary point) at around \( y^*=1000 \) with minimum value of 0, as shown by the following graph.

\textbf{At }\tilde{\mu}_3=25.6, \text{ there is another extreme point (maximum point), approximately at } 2000 \text{ with the maximum value of approximately } 8331.30.

There is a slight difference between Sections 2 and 3. In Section 2 there are two values of \( \mu_3 \) at which extreme point coincides with stationary point. In Section 3, there is only one value of \( \mu_3 \) at which extreme point coincides with stationary point. This property can be easily checked in the following way. The following data shows the combination of \( \mu_3 \) and the maximal value for \( f2[\mu_3] \) in Section 3. In this data, as \( \mu_3 \) increases the maximal value of \( f2[\mu_3] \) decreases quite slightly, in contrast to the case for Section 2.
The following program produces the graph of the combination of \( \mu_1 \) and the stationary points corresponding to each \( \mathrm{ff2}[\mu_3] \). In the figure point C corresponds to the combination of \( \mu_3 \) and the corresponding stationary points of \( \mathrm{ff2}[\mu_3] \).

Note that the lower part from point C indicates the stable stationary points while the higher part from point C indicates the unstable stationary points, while there is no such point B in Section 2 that is also an inflection point. Note furthermore that on the curve BC stationary points show local instability.
In this section, there is only one discontinuous point: $\bar{\mu}_3$, a small deviation of $\mu_3$ from which creates drastic deviation of stationary CO$_2$ level $Y^*$. 

4. DYNAMIC SYSTEM, INSTABILITY AND PIGOUVIAN TAX: LONG-RUN GENERAL EQUILIBRIUM DYNAMICS (LGED): Different Externality Function Case
CONCLUSIONS

In this paper, a dynamic process on CO₂ was constructed under the general equilibrium framework in order to examine the property of the stationary points of the dynamic process. In this primitive economic model there are two production sectors: agriculture sector and energy sector; and one (aggregate) household. Agriculture sector produces food utilizing labor and energy with CO₂ as external factor, while the energy industry requires only labor to produce energy. Household consumes food and energy, supplying labor. At each time, production and consumption are conducted with CO₂ given. This economic model must be constructed in the dynamic system, since the production and consumption of energy raises CO₂ at each time, while there are two mechanisms which reduces CO₂ at each time. One is the photosynthetic function of the trees and farm outputs and so on, and the other is a function of the sea as the greatest repository of carbon. Thus, CO₂ is subject to countervailing factors as time elapses: one is the enhancing factor exhibited by the combustion of fossil fuels through the human activity and the other are reducing factors just mentioned. In this way, this paper developed a primitive economic model to explore the variation of CO₂ through human activity.

In this paper, the externality function on the food production, f[Y], where Y is the level of CO₂ in the atmosphere, was assumed. In view of the photosynthesis, f[Y] is an increasing function of Y: 0≤Y≤Ȳ, while it is a decreasing function of Y≥Ȳ in view of the greenhouse effect. It was shown, first, by simulations, that the stationary points of the dynamic process consist of at most 3 points, depending on the positive parametric coefficient, which indicates the effect of the photosynthesis on the reduction of Y. When f[Y] is specified by transforming the Sin function, it was shown that there are two discontinuous points of the parameter, whose small deviation creates discontinuous movement of the stationary value of Y. Furthermore, when f[Y] is specified by transforming the Normal Distribution function, it was shown that there is one discontinuous point of the parameter, whose small deviation creates discontinuous movement of the stationary value of Y.

It was also shown by simulations that when there are three stationary points for some parameter, explained above, the Pigouvian taxation can reduce all the stationary points of Y, while for the smallest stationary point the utility level of household declines by the taxation. This is quite natural, since CO₂ in this case is “external economy” factor, not “external diseconomy”.

REFERENCES


