Public Good Provision: Lindahl Tax, Income Tax, Commodity Tax, and Poll Tax, II

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Introduction

There are two traditions in the argument on taxation. On the one hand, the argument on taxation is constructed in such a way that which taxation is desirable in order to impose a tax, income taxation or commodity taxation, without paying any attention on why the tax is necessary. On the other hand, it is also a tradition to assert that government imposes a tax in order to provide public goods. This paper integrates the above two traditions by constructing a primitive general equilibrium model which incorporates a public good, asking which taxation is desirable in order to impose tax, income taxation or commodity taxation to provide the public good. Formally, first, we utilize the Lindahl mechanism to compute a Pareto-optimal public good level. The burden-sharing in this Lindahl mechanism may be regarded as a tax on the society members, while utilizing pseudo-market mechanism. We call it Lindahl tax. We proceed to the computation of the rate of income tax in order to sustain the optimal public good level, and compare the Lindahl tax and income tax. Then, we compute the rate of (proportional) commodity tax in order to sustain the optimal public good level, and compare the Lindahl tax, income tax and commodity tax. Finally, we examine the poll tax. In the previous paper, Fukiharu [2012], the general equilibrium model was constructed, in which there was no poll tax to sustain the above public good level. In this paper, modifying the parameters in Fukiharu [2012], we examine if there is poll tax to sustain the above public good level.

I: Economy with Public Good and Walras - Lindahl Mechanism

We start with constructing a primitive general equilibrium model which incorporates a public good. The production and utility functions are specified by particular functions, and their parameters are also specified. Utilizing Walras-Lindahl mechanism, the optimal level of public good, as well as burden-sharing of each member of the society, are derived.

- Production of country A

Country A is under national isolation. She has three sectors of production, which produces 3 goods, utilizing labor, $L_i$, and capital, $K_i$: $y$ stands for the output of sector 1, $x$ stands for the output of sector 2, and $z$ stands for the output of sector 3 ($i=1, 2, 3$), where $y$ and $x$ are private goods and $z$ is a public good. Production function of sector 1, $y=f_{1A}$, is assumed as follows, with $a_1+b_1<1$: decreasing returns to scale.
\begin{verbatim}
In[1]:= f = L^a * K^b; c1 = {a -> 2/3, b -> 1/8, L -> L1, K -> K1};
f1 = f /. c1
Out[1]= K1^{1/8} L1^{2/3}

Production function of sector 2, \( x = f_2 \), is assumed as follows, with \( a_2 + b_2 < 1 \): decreasing returns to scale.

\begin{verbatim}
In[2]:= c2 = {a -> 1/2, b -> 1/3, L -> L2, K -> K2}; f2 = f /. c2
Out[2]= K2^{1/3} \sqrt{L2}
\end{verbatim}

Production function of sector 3, \( z = f_3 \), is assumed as follows, with \( a_3 + b_3 = 1 \): constant returns to scale.

\begin{verbatim}
In[3]:= c3 = {a -> 3/5, b -> 2/5, L -> L3, K -> K3}; f3 = f /. c3
Out[3]= K3^{2/5} L3^{3/5}
\end{verbatim}

From the profit maximization of the sector 1, demand for labor, \( L_1 \), and demand for capital, \( K_1 \), are computed as in what follows, where \( p_y \) stands for the price of the consumption good, \( y \), \( w_L \), wage rate of labor, and \( w_K \), rental price of capital.

\begin{verbatim}
In[4]:= pil = p_y * f1 - wL * L1 - wK * K1;
soll =
PowerExpand[Solve[{D[pil, L1] == 0, D[pil, K1] == 0},
{K1, L1}]] [[1]]
Out[4]= \begin{align*}
K1 & \rightarrow p_y^{24/5} / \left(54 \times 2^{3/5} 3^{1/5} wK^{8/5} wL^{16/5}\right), \\
L1 & \rightarrow \frac{4 \times 2^{2/5} p_y^{24/5}}{81 \times 3^{1/5} wK^{3/5} wL^{21/5}}
\end{align*}
\end{verbatim}

\begin{verbatim}
In[5]:= demand$L1 = L1 /. soll; demand$K1 = K1 /. soll;
\end{verbatim}

Thus, supply function of \( y \), \( y_A^S \), is computed as follows, with \( p_y \), \( w_L \), and \( w_K \), as parameters.

\begin{verbatim}
In[6]:= supply$A$y = PowerExpand[f1 /. soll];
\end{verbatim}

Profit function of sector 1, \( \pi_1 \), is computed as follows, with \( p_y \), \( w_L \), and \( w_K \), as parameters. This profit accrues to entrepreneur 1.

\begin{verbatim}
In[7]:= pi10 = PowerExpand[pil /. soll];
\end{verbatim}
\end{verbatim}
From the profit maximization of the sector 2, demand for labor, $L_2^A$, and demand for capital, $K_2^A$, as follows, where $p_x$ stands for the price of the consumption good, $x$.

\[
\text{In[8]:= } \pi_2 = px * f_2 - wL * L_2 - wK * K_2;
\]

\[\text{sol2 = PowerExpand[Solve[[D[\pi_2, L_2] == 0, D[\pi_2, K_2] == 0], }
\{K_2, L_2\]][[1]]\]

\[\text{Out[8]= } \left\{K_2 \rightarrow \frac{px^6}{216 wK^3 wL^3}, L_2 \rightarrow \frac{px^6}{144 wK^2 wL^4}\right\}\]

\[\text{In[9]:= } \text{demand}\$L_2 = L_2 /. \text{sol2}; \text{demand}\$K_2 = K_2 /. \text{sol2};\]

Thus, supply function of $x$, $x_A^S$, is computed as follows, with $p_x$, $w_L$, and $w_K$, as parameters.

\[\text{In[10]:= } \text{supply}\$A$x = \text{PowerExpand}[f_2 /. \text{sol2}];\]

Profit function of sector 2, $\pi_2^A$, is computed as follows, with $p_x$, $w_L$, and $w_K$, as parameters. This profit accrues to entrepreneur 2.

\[\text{In[11]:= } \pi_20 = \text{PowerExpand}[\pi_2 /. \text{sol2}];\]

The sector 3 produces a public good, $z$, under constant returns to scale, so that demand for capital, $K_3^A$, and the one for labor, $L_3^A$, is derived by the minimizing cost, given output level $z$, as in what follows.

\[\text{In[12]:= } \text{sol3 = Solve[[f_3 == z, D[f_3, K_3] / D[f_3, L_3] == wK / wL], } \{K_3, L_3\}] [[2]]\]

\[\text{Out[12]= } \left\{K_3 \rightarrow \frac{\left(\frac{2}{3}\right)^{3/5} wL^{3/5} z}{wK^{3/5}}, L_3 \rightarrow \frac{\left(\frac{3}{2}\right)^{2/5} wK^{2/5} z}{wL^{2/5}}\right\}\]

\[\text{In[13]:= } \text{demand}\$L_3 = L_3 / . \text{sol3}; \text{demand}\$K_3 = K_3 / . \text{sol3};\]

The homogeneity of degree one requires profit to be zero at the general equilibrium, so that $p_z z = w_L L_3^A + w_K K_3^A$ must hold at GE and $p_z$ must be equal to the following when $w_L = 1$. 


\textbf{Consumption of country A}

We proceed to the demand side of country A. She is endowed with \( L_{cA} = 100 \) and \( K_{cA} = 50 \).

\textbf{In [15]:} \hspace{1em} \texttt{LeA = 100; KeA = 50;}

The aggregate workers possesses (2/3) of \( L_{cA} \) and (1/5) of \( K_{cA} \), while the aggregate capitalists possesses (1/3) of \( L_{cA} \) and (4/5) of \( K_{cA} \). All the agents in this paper: workers, capitalists, and 2 entrepreneurs, have the same CES utility function, \( u[y, x, z] = (d_1 y^a + d_2 x^a + d_3 z^a)^{1/a} \) which is specified as \( u[y, x, z] = (y^{1/2} + x^{1/2} + (1/100) z^{1/2})^2 \); i.e. \( a = 1/2, d_1 = d_2 = 1 \), and \( d_3 = 1/100 \).

\textbf{In [16]:} \hspace{1em} \texttt{uA = (x}^(1/2) + y}^(1/2) + (1/100) z}^(1/2) ) ^2

\textbf{Out [16]}:

\[
\left( \sqrt{x} + \sqrt{y} + \frac{\sqrt{z}}{100} \right)^2
\]

All the consumers maximize utility subject to income constraint:

\[
\max u[y, x, z] \text{ s.t. } p_x y + p_y x + \theta_j p_z z = m_j \ (j = L, K, 1, 2) \quad (1)
\]

where \( m_j \) is income and \( \theta_j \) is the burden share of the household \( j \) \((j = L, K, 1, 2, 3)\).

Worker (household \( L \))’s income, \( m_L \), consists of (2/3)\( w_L L_{cA} \) + (1/5) \( w_K K_{cA} \). It is assumed that they supply (2/3)\( L_{cA} \) for labor supply and (1/5)\( K_{cA} \) for capital supply. Capitalist (household \( K \))’s income, \( m_L \), consists of (1/3)\( L_{cA} \) and (4/5) of \( K_{cA} \). It is assumed that they supply (1/3)\( L_{cA} \) for labor supply and (4/5)\( K_{cA} \) for capital supply. Entrepreneur 1 (household 1)’s income, \( m_1 \), consists of profit for the sector 1, \( \pi_{1A} \). Finally, entrepreneur 2 (household 2)’s income, \( m_2 \), consists of profit for the sector 2, \( \pi_{2A} \).

Demand function of workers for commodity \( y \), \( y_L^D \), that for commodity \( x \), \( x_L^D \), that for commodity \( z \), \( z_L^D \), demand function of capitalists for commodity \( y \), \( y_K^D \), that for commodity \( x \), \( x_K^D \), that for commodity \( z \), \( z_K^D \), demand function of entrepreneur 1 for commodity \( y \), \( y_{E1}^D \), that for commodity \( x \), \( x_{E1}^D \), that for commodity \( z \), \( z_{E1}^D \), and demand function of entrepreneur 2 for commodity \( y \), \( y_{E2}^D \), that for commodity \( x \), \( x_{E2}^D \), that for commodity \( z \), \( z_{E2}^D \), are derived as in what follows.
\textbf{In[17]} = 
\texttt{sol3A = Solve}\{\texttt{[D[uA, x] / D[uA, y] == px / py,}} \\texttt{D[uA, z] / D[uA, y] == tj * pz / py,}} \\texttt{px * x + py * y + tj * pz * z == m]}, \{x, y, z\}\}\texttt{[[1]]}

\textbf{Out[17]} = \{x \to (10000 m \texttt{px pz tj}) /} \\texttt{(px (px py + 10000 px pz tj + 10000 py pz tj))}, y \to \{(10000 m \texttt{px pz tj}) /} \\texttt{(py (px py + 10000 px pz tj + 10000 py pz tj))}, z \to \{(m \texttt{px py}) / (pz tj (px py + 10000 px pz tj + 10000 py pz tj))\}\}

\textbf{In[18]} = 
\texttt{demand$L$x = x / . sol3A / . \{m \to wL * (2 / 3) * \texttt{LeA + wK * (1 / 5) * KeA,}} \texttt{tj \to tL}\}; \texttt{demand$L$y = y / . sol3A / . \{m \to wL * (2 / 3) * \texttt{LeA + wK * (1 / 5) * KeA,}} \texttt{tj \to tL}\}; \texttt{demand$L$z = z / . sol3A / . \{m \to wL * (2 / 3) * \texttt{LeA + wK * (1 / 5) * KeA,}} \texttt{tj \to tL}\}; \texttt{demand$K$x = x / . sol3A / . \{m \to wL * (1 / 3) * \texttt{LeA + wK * (4 / 5) * KeA,}} \texttt{tj \to tK}\}; \texttt{demand$K$y = y / . sol3A / . \{m \to wL * (1 / 3) * \texttt{LeA + wK * (4 / 5) * KeA,}} \texttt{tj \to tK}\}; \texttt{demand$K$z = z / . sol3A / . \{m \to wL * (1 / 3) * \texttt{LeA + wK * (4 / 5) * KeA,}} \texttt{tj \to tK}\}; \texttt{demand$E1$x = x / . sol3A / . \{m \to \texttt{pi10, tj \to t1}\}; demand$E2$x = x / . sol3A / . \{m \to \texttt{pi20, tj \to t2}\}; demand$E1$y = y / . sol3A / . \{m \to \texttt{pi10, tj \to t1}\}; demand$E2$y = y / . sol3A / . \{m \to \texttt{pi20, tj \to t2}\}; demand$E1$z = z / . sol3A / . \{m \to \texttt{pi10, tj \to t1}\}; demand$E2$z = z / . sol3A / . \{m \to \texttt{pi20, tj \to t2}\}; {\texttt{demand$L$y, demand$L$x, demand$L$z, demand$K$y, demand$K$x, demand$K$z, demand$E1$y, demand$E1$x, demand$E1$z, demand$E2$y, demand$E2$x, demand$E2$z}}; \}

Country A’s demand for commodity \textit{y}, \textit{y}_A^D, is the sum of \textit{y}_L^D, \textit{y}_K^D, \textit{y}_{E1}^D, \textit{y}_{E2}^D.
In[19]:= demand$A$y = Simplify[demand$L$y + demand$K$y + demand$E1$y + demand$E2$y];

Country A's demand for commodity $x$, $x_A^D$, is the sum of $x_L^D$, $x_K^D$, $x_{E1}^D$, $x_{E2}^D$.

In[20]:= demand$A$x = Simplify[demand$L$x + demand$K$x + demand$E1$x + demand$E2$x];

Country A’s consumption of commodity $z$ is determined by the Lindahl mechanism: $z_A^D = z_L^D = z_K^D = z_{E1}^D = z_{E2}^D$ for suitable selection of $\theta_j$.

In[21]:= demand$L$z == demand$K$z == demand$E1$z == demand$E2$z;

- “GE with Public good”

General equilibrium for country A with public good, “GE with public good”, is defined by

\[
\begin{align*}
  y_A^D &= y_A^S, \\
  x_A^D &= x_A^S, \\
  z_{AL}^D &= z_{AK}^D = z_{AE1}^D = z_{AE2}^D = z, \\
  L_1^A + L_2^A + L_3^A &= L_eA, \\
  K_1^A + K_2^A + K_3^A &= K_eA.
\end{align*}
\]

From the application of Newton method on (2), (3), (4) and (6) we compute the GE with public good as in what follows.

In[22]:= check100 = 
\{supply$A$x == demand$A$x, supply$A$y == demand$A$y, 
  demand$K1$ + demand$K2$ + demand$K3$ == KeA, 
  demand$L$z == 
\{(demand$L$z + demand$K$z + demand$E1$z + demand$E2$z) / 4, demand$K$z == 
\{demand$L$z + demand$K$z + demand$E1$z + demand$E2$z) / 4, demand$E1$z == 
\{demand$L$z + demand$K$z + demand$E1$z + demand$E2$z) / 4, z == demand$E2$z) / . 
\{wL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow hom\};
In[23]:= \[\text{sol7} = \text{FindRoot}\left[\text{check100}, \{px, 7\}, \{py, 6\}, \{wK, 0.75\}, \{t1, 0.25\}, \{z, 1\}, \{tL, 0.25\}, \{tK, 0.25\}, \text{AccuracyGoal} \to 30, \text{WorkingPrecision} \to 20, \text{MaxIterations} \to 1000\right]\]

Out[23]= \{px \to 3.9848296356837408303, py \to 4.1581708016145142109, wK \to 0.79596080172148511116, t1 \to 0.16930458013370953559, z \to 0.038835026401740385141, tL \to 0.3494502086076803549, tK \to 0.32656092211915612127\}

It is confirmed that (5) is satisfied.

In[24]:= \[\text{demand}\_L1 + \text{demand}\_L2 + \text{demand}\_L3 - \text{LeA} \/. \{\text{wL} \to 1\} \/. \text{sol7}\]

Out[24]= 0. \times 10^{-18}

Finally, it is confirmed that (4) is satisfied.

In[25]:= \{\text{demand}\_L\_z == \text{demand}\_K\_z, \text{demand}\_K\_z == \text{demand}\_E1\_z, \\
\text{demand}\_E1\_z == \text{demand}\_E2\_z, \\
\text{demand}\_E2\_z == \text{demand}\_L\_z\} \/. \{\text{wL} \to 1, \text{t2} \to 1 - \text{t1} - \text{tL} - \text{tK}, \text{pz} \to \text{hom}\} \/. \text{sol7}\]

Out[25]= \{\text{True, True, True, True}\}

This “GE with public good” is also derived by the following Walras-Lindahl differential equations. \(\theta_j\)

\[
\begin{align*}
\frac{dp_y(t)}{dt} &= y_A D - y_A S \\
\frac{dp_x(t)}{dt} &= x_A D - x_A S \\
\frac{dw_K(t)}{dt} &= K_1 A D + K_2 A D + K_3 A D - K_c A \\
\frac{d\theta_1(t)}{dt} &= \frac{z_L D - (z_L D + z_K D + z_{E1} D + z_{E2} D)/4}{4} \\
\frac{d\theta_2(t)}{dt} &= \frac{z_K D - (z_L D + z_K D + z_{E1} D + z_{E2} D)/4}{4} \\
\frac{d\theta_3(t)}{dt} &= \frac{z_{E1} D - (z_L D + z_K D + z_{E1} D + z_{E2} D)/4}{4} \\
\frac{dz(t)}{dt} &= z_{E2} D - z(t)
\end{align*}
\]
Starting from the initial values: $p_x(0)=1$, $p_y(0)=1$, $w_k(0)=1$, $\theta_1(0)=0.1$, $\theta_k(0)=0.1$, $\theta_1(t)=0.1$, and $z(t)=1$; the trajectories on the differential equations: $p_y(t)$, $p_x(t)$, $w_k(t)$, $\theta_1(t)$, $\theta_k(t)$, $\theta_1(t)$, and $z(t)$; converge to the “GE with public good” as $t$ approaches 1000.
The “GE with public good” income for the (aggregate) workers before and after the deduction of the burden of consuming public good, are computed as follows.

\[
\text{income}^{A\,L}_{0A} = w_L \cdot (2/3) \cdot LeA + w_K \cdot (1/5) \cdot KeA /.
\text{income}^{A\,L}_{0B} = (w_L \cdot (2/3) \cdot LeA + w_K \cdot (1/5) \cdot KeA - t_j \cdot pz \cdot z) /.
\]

Its utility level at the GE with public good is computed as in what follows.

The workers’ consumption of commodities are computed as in what follows.

The “GE with public good” income for the (aggregate) capitalists before and after the deduction of the burden of consuming public good, are computed as follows.
In[33]:= \{income\$A\$K0A =
        \{wL \* (1 / 3) \* LeA + wK \* (4 / 5) \* KeA /.
        \{wL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow hom\} /.
        \text{sol7},
        income\$A\$K0B =
        \{wL \* (1 / 3) \* LeA + wK \* (4 / 5) \* KeA - tj \* pz \* z /.
        \{tj \rightarrow tK, z \rightarrow \text{demand\$K\$z}\} /.
        \{wL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow hom\} /.
        \text{sol7}\}\}

Out[33]= \{65.171765402192737780, 65.149075650456816028\}

Its utility level at the GE with public good is computed as in what follows.

In[34]:= uA0K =
        uA /. \{x \rightarrow \text{demand\$K\$x}, y \rightarrow \text{demand\$K\$y}, z \rightarrow \text{demand\$K\$z}\} /.
        \{wL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow hom\} /.
        \text{sol7}

Out[34]= 32.03930507626174350

The capitalists’ consumption of commodities are computed as in what follows.

In[35]:= \{K\$x0A =
        \text{demand\$K\$x} /.
        \{wL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow hom\} /.
        \text{sol7},
        K\$y0A =
        \text{demand\$K\$y} /.
        \{wL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow hom\} /.
        \text{sol7}\}\}

Out[35]= \{8.348652142324575074, 7.667101881271604378\}

The “GE with public good” income for the entrepreneur 1 before and after the deduction of the burden of consuming public good, are computed as follows.

In[36]:= \{income\$A\$E10A =
        (pi10) /.
        \{tj \rightarrow t1, z \rightarrow \text{demand\$E1\$z}\} /.
        \{wL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow hom\} /.
        \text{sol7},
        income\$A\$E10B =
        (pi10 - tj \* pz \* z) /.
        \{tj \rightarrow t1, z \rightarrow \text{demand\$E1\$z}\} /.
        \{wL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow hom\} /.
        \text{sol7}\}\}

Out[36]= \{17.52301733516896377, 17.51125389799448563\}

Its utility level at the GE with public good is computed as in what follows.
In[37]:= \text{UA}\text{E}1 = \\
\text{UA} \div \{ x \rightarrow \text{demand}\text{E}1x, y \rightarrow \text{demand}\text{E}1y, \\
z \rightarrow \text{demand}\text{E}1z \} \div \{ wL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow \text{hom} \} \div \text{sol7}

Out[37]= 8.6173370377122898

The entrepreneur 1’s consumption of commodities are computed as in what follows.

In[38]:= \{ E1\text{x0A} = \\
\text{demand}\text{E}1x \div \{ wL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow \text{hom} \} \div \text{sol7}, \\
E1\text{y0A} = \\
\text{demand}\text{E}1y \div \{ wL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow \text{hom} \} \div \text{sol7}

Out[38]= \{2.244012918228658360, 2.060820761680243735\}

The “GE with public good” income for the entrepreneur 2 before and after the deduction of the burden of consuming public good, are computed as follows.

In[39]:= \{ income\text{A}\text{E}20A = \\
(pi20 \div \{ tj \rightarrow t2, z \rightarrow \text{demand}\text{E}2z \} \div \text{sol7} \div wL \rightarrow 1, \\
income\text{A}\text{E}20B = \\
(pi20 \div tj \times pz \times z) \div \{ tj \rightarrow t2, z \rightarrow \text{demand}\text{E}2z \} \div \\
\{ wL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow \text{hom} \} \div \text{sol7}\}


Its utility level at the GE with public good is computed as in what follows.

In[40]:= \text{UA}\text{E}2 = \\
\text{UA} \div \{ x \rightarrow \text{demand}\text{E}2x, y \rightarrow \text{demand}\text{E}2y, \\
z \rightarrow \text{demand}\text{E}2z \} \div \{ wL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow \text{hom} \} \div \text{sol7}

Out[40]= 7.19420848604340194

The entrepreneur 2’s consumption of commodities are computed as in what follows.
Using the “modified GE with public good” incomes for 4 agents, the Gini coefficient before the deduction of the burden of consuming public good, $gini1$, is computed.

$$giniA[y_] := Module[{z1, z2}, z1 = Sort[y]; z2 = (y[[1]] + y[[2]] + y[[3]] + y[[4]]) / 4; 1 + (1/4) - 2 (1*\ y[[4]] + 2*\ y[[3]] + 3*\ y[[2]] + 4*\ y[[1]]) / (4^2*\ z2); giniA[{incomeA$A$L0A, incomeA$A$K0A, incomeA$A$E10A, incomeA$A$E20A}]]$$

$$Out[43]= 0.3309739379674046$$

Meanwhile the Gini coefficient after the deduction of the burden of consuming public good is computed as in what follows.

$$giniA[{incomeA$A$L0B, incomeA$A$K0B, incomeA$A$E10B, incomeA$A$E20B}]$$

$$Out[44]= 0.3310327896068688795$$

In this specified case, the income distribution inequality of country A is more unequal when the burden of consuming public good is deducted for each member.

The optimal public good level is computed as in what follows.

$$\{\ text{demand} L z, \ text{demand} K z, \ text{demandE1} z, \ text{demandE2} z\} / . \{\ text{pz} \rightarrow \ hom, \ text{t2} \rightarrow 1 - \ t1 - \ tL - \ tK\} / . \text{sol7} / . \text{wL} \rightarrow 1$$

$$Out[45]= \{0.03883502640174038514, 0.03883502640174038514, 0.03883502640174038514, 0.03883502640174038514\}$$

The price of public good, $p_{z0}$, is computed as in what follows.
II : Income Tax Provision of Public Good

In this section, we examine if it is possible to use income taxation instead to Lindahl taxation in order to achieve the optimum public good level, \( z_0 = 0.038835 \). Since \( z_0 \) is provided for each member of the society, the utility function becomes the following one \( u(y, x, z_0) \).

\[
\text{max } u[y, x, z_0] \text{ s.t. } p_y y + p_x x = (1-\tau_y) m_j \quad (j = L, K, 1, 2)
\]

(7)

where \( m_j \) is income and \( \tau_y \) is the rate of income tax \( (j = L, K, 1, 2, 3) \). Worker (household \( L \))’s pre-tax income, \( m_L \), consists of \( (2/3)w_L L_{cA} + (1/5) w_K K_{cA} \). It is assumed that they supply \( (2/3)L_{cA} \) for labor supply and \( (1/5)K_{cA} \) for capital supply. Capitalist (household \( K \))’s pre-tax income, \( m_L \), consists of \( (1/3) L_{cA} \) and \( (4/5) K_{cA} \). It is assumed that they supply \( (1/3)L_{cA} \) for labor supply and \( (4/5)K_{cA} \) for capital supply. Entrepreneur 1 (household 1)’s pre-tax income, \( m_1 \), consists of profit for the sector 1, \( \pi_{1A} \). Finally, entrepreneur 2 (household 2)’s pre-tax income, \( m_2 \), consists of profit for the sector 2, \( \pi_{2A} \).

Demand function of workers for commodity \( y \), \( y_L^D \), that for commodity \( x \), \( x_L^D \), the demand function of capitalists for commodity \( y \), \( y_K^D \), that for commodity \( x \), \( x_K^D \), the demand function of entrepreneur 1 for commodity \( y \), \( y_{E1}^D \), that for commodity \( x \), \( x_{E1}^D \), and the demand function of entrepreneur 2 for commodity \( y \), \( y_{E2}^D \), that for commodity \( x \), \( x_{E2}^D \), are derived as in what follows.
\textbf{In[49]}:=  
\texttt{sol3B = Solve}\{D[uA0, x] / D[uA0, y] = px / py,} \n\quad \texttt{px \times x + py \times y = (1 - ty) \times m}, \{x, y}\}\}[[1]]  
\textbf{Out[49]}= \{x \rightarrow \frac{m \times px - m \times py \times ty}{px \times (px + py)}, y \rightarrow \frac{m \times px - m \times px \times ty}{py \times (px + py)}\}

\textbf{In[50]}:=  
\texttt{demand}L\texttt{x}\texttt{B} = \texttt{x} /. \texttt{sol3B} /. \{m \rightarrow wL \times (2/3) \times \texttt{LeA} + wK \times (1/5) \times \texttt{KeA}\};  
\texttt{demand}L\texttt{y}\texttt{B} = \texttt{y} /. \texttt{sol3B} /. \{m \rightarrow wL \times (2/3) \times \texttt{LeA} + wK \times (1/5) \times \texttt{KeA}\};  
\texttt{demand}K\texttt{x}\texttt{B} = \texttt{x} /. \texttt{sol3B} /. \{m \rightarrow wL \times (1/3) \times \texttt{LeA} + wK \times (4/5) \times \texttt{KeA}\};  
\texttt{demand}K\texttt{y}\texttt{B} = \texttt{y} /. \texttt{sol3B} /. \{m \rightarrow wL \times (1/3) \times \texttt{LeA} + wK \times (4/5) \times \texttt{KeA}\};  
\texttt{demand}E1\texttt{x}\texttt{B} = \texttt{x} /. \texttt{sol3B} /. \{m \rightarrow \texttt{pi10}\};  
\texttt{demand}E2\texttt{x}\texttt{B} = \texttt{x} /. \texttt{sol3B} /. \{m \rightarrow \texttt{pi20}\};  
\texttt{demand}E1\texttt{y}\texttt{B} = \texttt{y} /. \texttt{sol3B} /. \{m \rightarrow \texttt{pi10}\};  
\texttt{demand}E2\texttt{y}\texttt{B} = \texttt{y} /. \texttt{sol3B} /. \{m \rightarrow \texttt{pi20}\};

\textbf{In[51]}:=  
\texttt{demand}A\texttt{y}\texttt{B} =  
\texttt{Simplify[\{demand}L\texttt{y}\texttt{B} + demand\texttt{K}\texttt{y}\texttt{B} + demand\texttt{E1}\texttt{y}\texttt{B} + \texttt{demand}\texttt{E2}\texttt{y}\texttt{B}\};}

\textbf{In[52]}:=  
\texttt{demand}A\texttt{x}\texttt{B} =  
\texttt{Simplify[\{demand}L\texttt{x}\texttt{B} + demand\texttt{K}\texttt{x}\texttt{B} + demand\texttt{E1}\texttt{x}\texttt{B} + \texttt{demand}\texttt{E2}\texttt{x}\texttt{B}\};}

General equilibrium for country A with public good when income taxation is adopted instead of Lindahl taxation, “GE with public good on income tax”, is defined by

\begin{align*}
y_A^D &= y_A^S, & (8) \\
x_A^D &= x_A^S, & (9) \\
p_z^A &= \tau_y (w_L L_{cA} + w_K K_{cA} + \pi_1 + \pi_2) & (10) \\
L_{1A}^D + L_{2A}^D + L_{3A}^D &= L_{cA}, & (11) \\
K_{1A}^D + K_{2A}^D + K_{3A}^D &= K_{cA}. & (12)
\end{align*}

From the application of Newton method on (8), (9), (10) and (12) we compute the GE with public good on income tax as in what follows.
In[53]:= check200 = 
   {supply$A$x == demand$A$x$y$, supply$A$y == demand$A$y$xB$, 
    demand$A$1 + demand$A$2 + demand$A$3 == KeA, 
    demand$L$1 + demand$L$2 + demand$L$3 == LeA, 
    $pz$ * z00 = ty ($wL$ * LeA + $wK$ * KeA + pi10 + pi20)} /. 
   {wL \rightarrow 1, z \rightarrow z00};

In[54]:= sol8 = FindRoot[check200, {px, 4}, {py, 4}, {pz, 2}, 
   {wK, 1}, {ty, 0.001}, WorkingPrecision \rightarrow 20, 
   AccuracyGoal \rightarrow 30, MaxIterations \rightarrow 1000]

Out[54]= {px \rightarrow 3.9848296356837408303, py \rightarrow 4.1581708016145142109, 
   pz \rightarrow 1.7891301135403671253, wK \rightarrow 0.79596080172148511116, 
   ty \rightarrow 0.00040407797945003617712}

Note that we have exactly the same equilibrium prices for commodities and inputs.

In[55]:= sol7

Out[55]= {px \rightarrow 3.9848296356837408303, 
   py \rightarrow 4.1581708016145142109, wK \rightarrow 0.79596080172148511116, 
   tl \rightarrow 0.16930458013370953559, z \rightarrow 0.038835026401740385141, 
   tl \rightarrow 0.34945020860776803549, tk \rightarrow 0.32656092211915612127}

In[56]:= check200 /. sol8

Out[56]= {True, True, True, True, True}

Profit for the sector 3 is (almost) zero as shown in what follows.

In[57]= (pz * z00 - (wL * demand$L$3 + wK * demand$K$3)) /. sol8 /. 
   wL \rightarrow 1 /. z \rightarrow z00

Out[57]= 0. \times 10^{-21}

Furthermore, the price of public good in this section is exactly the same one as in the 
previous section.

In[58]= pzO

Out[58]= 1.78913011354036712528

The “GE with public good on income tax” income for the (aggregate) workers is 
computed as follows.
In[59]:= \textcolor{black}{\texttt{incomeA}L1B =} \\
\hspace{1.2cm} (1 - \texttt{ty}) \ast (\texttt{wL} \ast (2 / 3) \ast \texttt{LeA} + \texttt{wK} \ast (1 / 5) \ast \texttt{KeA}) / . \texttt{wL} \rightarrow 1 / . \texttt{sol8} \\
Out[59]= 74.5961198499627473162 \\

It is smaller than the income in “GE with public good” after the deduction of the burden of consuming public good.

In[60]:= \textcolor{black}{\texttt{incomeA}L0B} \\
Out[60]= 74.6019945635721955316 \\

The “GE with public good on income tax” utility level for the (aggregate) workers is computed as follows.

In[61]:= \textcolor{black}{\texttt{uA1L =} } \\
\hspace{1.2cm} \texttt{uA} / . \{\texttt{x} \rightarrow \texttt{demandL}$x1B, \texttt{y} \rightarrow \texttt{demandL}$y1B, \texttt{z} \rightarrow \texttt{z00} \} / . \texttt{sol8} / . \texttt{wL} \rightarrow 1 \\
Out[61]= 36.68354270520764598 \\

It is lower than the utility in “GE with public good”.

In[62]:= \textcolor{black}{\texttt{uA0L}} \\
Out[62]= 36.68643072656694503 \\

The workers’ consumption of commodities are computed as in what follows.

In[63]:= \{\texttt{L}$x1A = \texttt{demandL}$x1B / . \{\texttt{z} \rightarrow \texttt{z00} \} / . \texttt{sol8} / . \texttt{wL} \rightarrow 1, \\
\hspace{1.2cm} \texttt{L}$y1A = \texttt{demandL}$y1B / . \{\texttt{z} \rightarrow \texttt{z00} \} / . \texttt{sol8} / . \texttt{wL} \rightarrow 1\} \\
Out[63]= \{9.5592615793332744, 8.778882050542203148\} \\

Compare this with the consumptions in the previous section.

In[64]:= \{\texttt{L}$x0A, \texttt{L}$y0A\} \\
Out[64]= \{9.560014405676163197, 8.779573419181239494\} \\

The “GE with public good on income tax” income for the (aggregate) capitalists is computed as follows.

In[65]:= \textcolor{black}{\texttt{incomeA}K1B =} \\
\hspace{1.2cm} (1 - \texttt{ty}) \ast (\texttt{wL} \ast (1 / 3) \ast \texttt{LeA} + \texttt{wK} \ast (4 / 5) \ast \texttt{KeA}) / . \texttt{wL} \rightarrow 1 / . \texttt{sol8} \\
Out[65]= 65.14543092723406989
It is smaller than the income in “GE with public good” after the deduction of the burden of consuming public good.

In[66]:= \text{income}\text{A}\text{K0B}  
\text{Out}[66]= 65.149075650456816028  

The “GE with public good on income tax” utility level for the (aggregate) capitalists is computed as follows.

In[67]:= \text{uA1K} = \text{uA} / \{ \text{x} \to \text{demand}\text{KxB}, \text{y} \to \text{demand}\text{KyB}, \text{z} \to \text{z00}\} / \text{sol8} / \text{wL} \to 1  
\text{Out}[67]= 32.03751328193042107  

It is lower than the utility in “GE with public good”.

In[68]:= \text{uA0K}  
\text{Out}[68]= 32.03930507626174350  

The capitalists’ consumption of commodities are computed as in what follows.

In[69]:= \{ \text{Kx1A} = \text{demand}\text{KxB} / \{ \text{z} \to \text{z00}\} / \text{sol8} / \text{wL} \to 1,  
\text{Ky1A} = \text{demand}\text{KyB} / \{ \text{z} \to \text{z00}\} / \text{sol8} / \text{wL} \to 1\}  
\text{Out}[69]= \{ 8.348185082338916905, 7.66672950177859064 \}  

Compare this with the consumptions in the previous section.

In[70]:= \{ \text{Kx0A}, \text{Ky0A} \}  
\text{Out}[70]= \{ 8.348652142324575074, 7.667101881271604378 \}  

The “GE with public good on income tax” income for the entrepreneur 1 is computed as follows.

In[71]:= \text{income}\text{A}\text{E11B} = (1 - \text{ty}) \ast \text{pi0} / \text{wL} \to 1 / \text{sol8}  
\text{Out}[71]= 17.51593666981703397  

It is greater than the income in “GE with public good” after the deduction of the burden of consuming public good.
The “GE with public good on income tax” utility level for the entrepreneur 1 is computed as follows.

\[
\text{uA1E1} = \text{uA} \cdot \{x \rightarrow \text{demandE1xB}, \ y \rightarrow \text{demandE1yB}, \ z \rightarrow z00\} \cdot \text{sol8} \cdot \text{wL} \rightarrow 1
\]

It is higher than the utility in “GE with public good”.

The entrepreneur 1’s consumption of commodities are computed as in what follows.

\[
\{E1x1A = \text{demandE1xB} \cdot \{z \rightarrow z00\} \cdot \text{sol8} \cdot \text{wL} \rightarrow 1, \\
E1y1A = \text{demandE1yB} \cdot \{z \rightarrow z00\} \cdot \text{sol8} \cdot \text{wL} \rightarrow 1\}
\]

Compare this with the consumptions in the previous section.

The “GE with public good on income tax” income for the entrepreneur 2 is computed as follows.

\[
\text{incomeA21B} = (1-ty) \cdot \text{pi20} \cdot \text{wL} \rightarrow 1 \cdot \text{sol8}
\]

It is greater than the income in “GE with public good” after the deduction of the burden of consuming public good.

The “GE with public good on income tax” utility level for the entrepreneur 2 is
computed as follows.

\[
\text{In[79]} := \text{uA1E2} = \\
\text{uA} / \cdot \{x \rightarrow \text{demand}$E2\$xB, y \rightarrow \text{demand}$E2\$yB, z \rightarrow z00\} \/. \text{sol8} \/. \text{wL} \rightarrow 1
\]

\[
\text{Out[79]} = 7.196587174464033883
\]

It is higher than the utility in “GE with public good”.

\[
\text{In[80]} := \text{uA0E2}
\]

\[
\text{Out[80]} = 7.19420848604340194
\]

The entrepreneur 2’s consumption of commodities are computed as in what follows.

\[
\text{In[81]} := \{E2$\text{x1A} = \text{demand}$E2$xB \/. \{z \rightarrow z00\} \/. \text{sol8} \/. \text{wL} \rightarrow 1, \\
E2$\text{y1A} = \text{demand}$E2$yB \/. \{z \rightarrow z00\} \/. \text{sol8} \/. \text{wL} \rightarrow 1\}
\]

\[
\text{Out[81]} = \{1.873803418403045445, 1.7208367104704320\}
\]

Compare this with the consumptions in the previous section.

\[
\text{In[82]} := \{E2$\text{x0A}, E2$\text{y0A}\}
\]

\[
\text{Out[82]} = \{1.873183614771320547, 1.720264465681936069\}
\]

From the viewpoint of Bentham-type utilitarian, the income taxation is more desirable than the Lindahl taxation, since the sum of utility for the former case is greater than the one for the latter case, as shown in what follows.

\[
\text{In[83]} := \{\text{uA0L} + \text{uA0K} + \text{uA0E1} + \text{uA0E2}, \text{uA1L} + \text{uA1K} + \text{uA1E1} + \text{uA1E2}\}
\]

\[
\text{Out[83]} = \{84.53727799264331946, 84.53727972251664468\}
\]

Furthermore, the income taxation is more desirable than the Lindahl taxation in the sense that the Gini coefficient for the former is smaller than the latter case, as shown in what follows.

\[
\text{In[84]} := \text{giniA}[\text{\{income}$A$\text{L1B, income}$A$\text{K1B, income}$A$\text{E11B}, \\
\text{income}$A$\text{E21B}\}]
\]

\[
\text{Out[84]} = 0.3309739379674684046
\]
In[85]:= giniA[{income$A$L0B, income$A$K0B, income$A$E10B, income$A$E20B}]

Out[85]= 0.33103278960688795

Note that the Gini coefficient for the income taxation is exactly the same as in the one for the incomes before the Lindahl taxation.

In[86]:= giniA[{income$A$L0A, income$A$K0A, income$A$E10A, income$A$E20A}]

Out[86]= 0.3309739379674684046

III: (Proportional) Commodity Tax Provision of Public Good

In this section, we examine if it is possible to use (proportional) commodity taxation instead to Lindahl taxation in order to achieve the optimum public good level, $z_0 = 0.038835$. Since $z_0$ is provided for each member of the society, the utility function becomes the following one $u(y, x, z_0)$.

In[87]:= uA0 = uA /. z -> z00

Out[87]= 

\[
\left(0.0019706604578602673413 + \sqrt{x} + \sqrt{y}\right)^2
\]

All the consumers maximize utility subject to income constraint:

\[
\max u(y, x, z_0) \text{ s.t. } (1+\tau_1) p_y y + (1+\tau_1) p_x x = m_j \quad (j = L, K, 1, 2) \quad (13)
\]

where $m_j$ is income and $\tau_1$ is the rate of commodity tax ($j = L, K, 1, 2, 3$). Worker (household $L$)'s income, $m_L$, consists of $(2/3)w_L L_{cA} + (1/5) w_K K_{cA}$. It is assumed that they supply $(2/3) L_{cA}$ for labor supply and $(1/5) K_{cA}$ for capital supply. Capitalist (household $K$)'s income, $m_L$, consists of $(1/3) L_{cA}$ and $(4/5) K_{cA}$. It is assumed that they supply $(1/3) L_{cA}$ for labor supply and $(4/5) K_{cA}$ for capital supply. Entrepreneur 1 (household 1)'s income, $m_1$, consists of profit for the sector 1, $\pi_{1A}$. Finally, entrepreneur 2 (household 2)'s income, $m_2$, consists of profit for the sector 2, $\pi_{2A}$.

Demand function of workers for commodity $y$, $y_L^D$, that for commodity $x$, $x_L^D$, the demand function of capitalists for commodity $y$, $y_K^D$, that for commodity $x$, $x_K^D$, the
demand function of entrepreneur 1 for commodity $y$, $y_{E1}^D$, that for commodity $x$, $x_{E1}^D$, and the demand function of entrepreneur 2 for commodity $y$, $y_{E2}^D$, that for commodity $x$, $x_{E2}^D$, are derived as in what follows.

In[88]:= \text{sol3C} = \\
\quad \text{Solve}[\{D[uA0, x] / D[uA0, y] = px / py,} \\
\quad \quad \quad \quad \quad (1 + t1) * px * x + (1 + t1) * py * y = m\}, \{x, y\}][[1]]

Out[88]= \begin{align*}
&x \to \frac{m \cdot py}{px (px + py) (1 + t1)}, \\
&y \to \frac{m \cdot px}{py (px + py) (1 + t1)}
\end{align*}

In[89]:= \text{demand$L$xC} = \\
\quad x /. \text{sol3C} /. \{m \to wL * (2 / 3) * LeA + wK * (1 / 5) * KeA\};
\text{demand$L$yC} = \\
\quad y /. \text{sol3C} /. \{m \to wL * (2 / 3) * LeA + wK * (1 / 5) * KeA\};
\text{demand$K$xC} = \\
\quad x /. \text{sol3C} /. \{m \to wL * (1 / 3) * LeA + wK * (4 / 5) * KeA\};
\text{demand$K$yC} = \\
\quad y /. \text{sol3C} /. \{m \to wL * (1 / 3) * LeA + wK * (4 / 5) * KeA\};
\text{demand$E1$xC} = x /. \text{sol3C} /. \{m \to \pi10\};
\text{demand$E1$yC} = x /. \text{sol3C} /. \{m \to \pi20\};
\text{demand$E2$xC} = y /. \text{sol3C} /. \{m \to \pi10\};
\text{demand$E2$yC} = y /. \text{sol3C} /. \{m \to \pi20\};

In[90]:= \text{demand$A$yC} = \\
\quad \text{Simplify}[\text{demand$L$yC} + \text{demand$K$yC} + \text{demand$E1$yC} + \text{demand$E2$yC}];

In[91]:= \text{demand$A$xC} = \\
\quad \text{Simplify}[\text{demand$L$xC} + \text{demand$K$xC} + \text{demand$E1$xC} + \text{demand$E2$xC}];

General equilibrium for country A with public good when (proportional) taxation is adopted instead of Lindahl taxation, “GE with public good on commodity tax”, is defined by

\begin{align*}
y^A_D &= y^A_S, \\
x^A_D &= x^A_S, \\
p_z &= \tau_1 p_y (y^L_D + y^K_D + y^E1_D + y^E2_D) + \\
&\quad \quad \tau_1 p_x (x^L_D + x^K_D + x^E1_D + x^E2_D) \\
L^A_D + L^2_A D + L^3_A D &= L_{cA},
\end{align*}

(14) (15) (16) (17)
General equilibrium for country A with public good when (proportional) taxation is adopted instead of Lindahl taxation, "GE with public good on commodity tax", is defined by

\[ y^D_A = y^S_A, \quad x^D_A = x^S_A, \quad p_z z_0^0 = t_1 x^C + t_1 p_y y^C + t_1 x^C + t_1 p_y y^C \]

From the application of Newton method on (14), (15), (16) and (18) we compute “the GE with public good on commodity tax” as in what follows.

In[92]:= check300 = 
{supply$A$x == demand$A$x, supply$A$y == demand$A$y, 
demand$K1$ + demand$K2$ + demand$K3$ == KeA, 
demand$L1$ + demand$L2$ + demand$L3$ == LeA, 
p_z z_00 == t_1 x^C + t_1 p_y y^C} /. 
{wL -> 1, z -> z_00};

In[93]:= sol9 = FindRoot[check300, {px, 15}, {py, 17}, {pz, 3}, 
{wK, 3}, {t_1, 0.2}, PrecisionGoal -> 30, 
WorkingPrecision -> 20, MaxIterations -> 1000]

Out[93]= {px -> 3.9848296356837408303, 
py -> 4.1581708016145142109, 
pz -> 1.7891301135403671253, wK -> 0.7959608017214851116, 
t_1 -> 0.0004042413017214851116}

Note that we have exactly the same equilibrium prices for commodities and inputs.

In[94]:= sol7

Out[94]= {px -> 3.9848296356837408303, 
py -> 4.1581708016145142109, wK -> 0.7959608017214851116, 
t_1 -> 0.169304580133705359, z -> 0.038835026401740385141, 
t_L -> 0.34945020860776803549, t_K -> 0.3265609221191561217}

In[95]:= check300 /. sol9

Out[95]= {True, True, True, True, True}

Profit for the sector 3 is zero as shown in what follows.

In[96]:= (p_z z_00 == w_L x^L3 + w_K x^K3) /. sol9 /. 
w_L -> 1 /. z -> z_00

Out[96]= True

On the one hand, workers’ income for “the GE with public good on commodity tax” is greater than the one for “the GE with public good on income tax”.

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In[97]:= \{income$\text{A}\$L0C =
(wL \ast (2 / 3) \ast LeA + wK \ast (1 / 5) \ast KeA) / wL \rightarrow 1 / . sol9,
income$\text{A}\$L0B\}

Out[97]= \{74.6262746838815177783, 74.6019945635721955316\}

On the other hand, workers’ commodity consumptions for “the GE with public good on commodity tax” are exactly the same as those for “the GE with public good on income tax” as shown in what follows.

In[98]:= \{x \rightarrow \text{demand$L}\$xC, y \rightarrow \text{demand$L}\$yC, z \rightarrow z00\} / . sol9 /.
wL \rightarrow 1

Out[98]= \{x \rightarrow 9.559261579333327444,
y \rightarrow 8.778882050542203148, z \rightarrow 0.038835026401740385141\}

In[99]:= \{x \rightarrow \text{demand$L}\$xB, y \rightarrow \text{demand$L}\$yB, z \rightarrow z00\} / . sol8 /.
wL \rightarrow 1

Out[99]= \{x \rightarrow 9.559261579333327444,
y \rightarrow 8.778882050542203148, z \rightarrow 0.038835026401740385141\}

Thus, workers’ utility level for “the GE with public good on commodity tax” is exactly the same as the one for “the GE with public good on income tax” as shown in what follows.

In[100]:= \{uA0LC =
ua / . \{x \rightarrow \text{demand$L}\$xC, y \rightarrow \text{demand$L}\$yC, z \rightarrow z00\} / .
sol9 / . wL \rightarrow 1, uA1L\}

Out[100]= \{36.68354270520764598, 36.68354270520764598\}

On the one hand, capitalists’ income for “the GE with public good on commodity tax” is greater than the one for “the GE with public good on income tax”.

In[101]:= \{income$\text{A}\$K0C =
(wL \ast (1 / 3) \ast LeA + wK \ast (4 / 5) \ast KeA) / wL \rightarrow 1 / . sol9,
income$\text{A}\$K0B\}

Out[101]= \{65.171765402192737780, 65.149075650456816028\}

On the other hand, capitalists’ consumptions of commodities for “the GE with public good on commodity tax” are exactly the same as those for “the GE with public good on income tax”.

On the other hand, capitalists’ consumption of commodities for “the GE with public good on commodity tax” are exactly the same as those for “the GE with public good on income tax”.
\text{In[102]} := \{x \rightarrow \text{demand}KxC, \ y \rightarrow \text{demand}KyC, \ z \rightarrow z00\} /\ . \text{sol9} /\ . \\
\text{wL} \rightarrow 1 \\
\text{Out[102]} = \{x \rightarrow 8.34815082238916905, \\
\ y \rightarrow 7.666672950177859064, \ z \rightarrow 0.038835026401740385141\} \\
\text{In[103]} := \{x \rightarrow \text{demand}KxB, \ y \rightarrow \text{demand}KyB, \ z \rightarrow z00\} /\ . \text{sol8} /\ . \\
\text{wL} \rightarrow 1 \\
\text{Out[103]} = \{x \rightarrow 8.34815082238916905, \\
\ y \rightarrow 7.666672950177859064, \ z \rightarrow 0.038835026401740385141\} \\

Thus, capitalists’ utility level for “the GE with public good on commodity tax” is exactly the same as the one for “the GE with public good on income tax” as shown in what follows.

\text{In[104]} := \{uA0K = \text{uA} /\ . \{x \rightarrow \text{demand}KxC, \ y \rightarrow \text{demand}KyC, \ z \rightarrow z00\} /\ . \\
\text{sol9} /\ . \text{wL} \rightarrow 1, \text{uA1K}\} \\
\text{Out[104]} = \{32.03751328193042107, 32.03751328193042107\} \\

On the one hand, entrepreneur 1’s income for “the GE with public good on commodity tax” is greater than the one for “the GE with public good on income tax”.

\text{In[105]} := \{\text{income}A$E10C = \text{pi10} /\ . \text{wL} \rightarrow 1 /\ . \text{sol9}, \text{income}A$E10B\} \\
\text{Out[105]} = \{17.52301733516896377, 17.51125389799448563\} \\

On the other hand, entrepreneur 1’s consumptions of commodities for “the GE with public good on commodity tax” are exactly the same as those for “the GE with public good on income tax”.

\text{In[106]} := \{x \rightarrow \text{demand}E1$xC, \ y \rightarrow \text{demand}E1$yC, \ z \rightarrow z00\} /\ . \text{sol9} /\ . \\
\text{wL} \rightarrow 1 \\
\text{Out[106]} = \{x \rightarrow 2.244613000925427384, \\
\ y \rightarrow 2.061371856047919145, \ z \rightarrow 0.038835026401740385141\} \\
\text{In[107]} := \{x \rightarrow \text{demand}E1$xB, \ y \rightarrow \text{demand}E1$yB, \ z \rightarrow z00\} /\ . \text{sol8} /\ . \\
\text{wL} \rightarrow 1 \\
\text{Out[107]} = \{x \rightarrow 2.244613000925427384, \\
\ y \rightarrow 2.061371856047919145, \ z \rightarrow 0.038835026401740385141\} \\

Thus, entrepreneur 1’s utility level for “the GE with public good on commodity tax” is
Thus, entrepreneur 1’s utility level for “the GE with public good on commodity tax” is exactly the same as the one for “the GE with public good on income tax” as shown in what follows..

\[
\text{In[108]:= } \{uA0E1C = uA / . \{x \rightarrow \text{demand$E1$xC}, \ y \rightarrow \text{demand$E1$yC}, \ z \rightarrow z00\} / . \text{sol9} / . \text{wL} \rightarrow 1, \ uA1E1\}
\]

\[
\text{Out[108]= } \{8.61963656091454375, 8.61963656091454375\}
\]

On the one hand, entrepreneur 2’s income for “the GE with public good on commodity tax” is greater than the one for “the GE with public good on income tax”.

\[
\text{In[109]:= } \{\text{income$A$E20C} = \text{pi20} / . \text{wL} \rightarrow 1 / . \text{sol9}, \ \text{income$A$E20B}\}
\]

\[
\text{Out[109]= } \{14.62821865944734086, 14.61747105347157406\}
\]

On the other hand, entrepreneur 2’s consumptions of commodities for “the GE with public good on commodity tax” are exactly the same as those for “the GE with public good on income tax”.

\[
\text{In[110]:= } \{x \rightarrow \text{demand$E2$xC}, \ y \rightarrow \text{demand$E2$yC}, \ z \rightarrow z00\} / . \text{sol9} / . \text{wL} \rightarrow 1
\]

\[
\text{Out[110]= } \{x \rightarrow 1.873803418403045445, \ y \rightarrow 1.720833671047042320, \ z \rightarrow 0.038835026401740385141\}
\]

\[
\text{In[111]:= } \{x \rightarrow \text{demand$E2$xB}, \ y \rightarrow \text{demand$E2$yB}, \ z \rightarrow z00\} / . \text{sol8} / . \text{wL} \rightarrow 1
\]

\[
\text{Out[111]= } \{x \rightarrow 1.873803418403045445, \ y \rightarrow 1.720833671047042320, \ z \rightarrow 0.038835026401740385141\}
\]

Thus, entrepreneur 2’s utility level for “the GE with public good on commodity tax” is exactly the same as the one for “the GE with public good on income tax” as shown in what follows.

\[
\text{In[112]:= } \{uA0E2C = uA / . \{x \rightarrow \text{demand$E2$xC}, \ y \rightarrow \text{demand$E2$yC}, \ z \rightarrow z00\} / . \text{sol9} / . \text{wL} \rightarrow 1, \ uA1E2\}
\]

\[
\text{Out[112]= } \{7.196587174464033883, 7.196587174464033883\}
\]

Thus, the conclusion for “the GE with public good on commodity tax” is exactly the same as the one for “the GE with public good on income tax”.
IV : Poll Tax Provision of Public Good

In this section, we examine if it is possible to use (proportional) commodity taxation instead to Lindahl taxation in order to achieve the optimum public good level, \( z_o = 0.038835 \). Since \( z_o \) is provided for each member of the society, the utility function becomes the following one \( u[y, x, z_o] \).

\[
\text{In[113]} = \text{uA0 = uA /. z -> z00}
\]

\[
\text{Out[113]} = \left( 0.00197066045786026734143 + \sqrt{x} + \sqrt{y} \right)^2
\]

All the consumers maximize utility subject to income constraint:

\[
\text{max } u[y, x, z_o] \text{ s.t. } p_y y + p_x x = m_j - T/4 \quad (j = L, K, 1, 2)
\]

where \( m_j \) is pre-tax income and \( T \) is the tax to sustain \( z_o \) \((j = L, K, 1, 2, 3)\). Worker (household \( L \))’s pre-tax income, \( m_L \), consists of \((2/3)w_L L_{cA} + (1/5)w_K K_{cA}\). It is assumed that they supply \((2/3)L_{cA}\) for labor supply and \((1/5)K_{cA}\) for capital supply. Capitalist (household \( K \))’s pre-tax income, \( m_L \), consists of \((1/3)\) of \( L_{cA} \) and \((4/5)\) of \( K_{cA} \). It is assumed that they supply \((1/3)L_{cA}\) for labor supply and \((4/5)K_{cA}\) for capital supply. Entrepreneur 1 (household 1)’s pre-tax income, \( m_1 \), consists of profit for the sector 1, \( \pi_1 A \). Finally, entrepreneur 2 (household 2)’s pre-tax income, \( m_2 \), consists of profit for the sector 2, \( \pi_2 A \).

Demand function of workers for commodity \( y \), \( y_L D \), that for commodity \( x \), \( x_L D \), the demand function of capitalists for commodity \( y \), \( y_K D \), that for commodity \( x \), \( x_K D \), the demand function of entrepreneur 1 for commodity \( y \), \( y_{E1} D \), that for commodity \( x \), \( x_{E1} D \), and the demand function of entrepreneur 2 for commodity \( y \), \( y_{E2} D \), that for commodity \( x \), \( x_{E2} D \), are derived as in what follows.

\[
\text{In[114]} = \text{sol3D = }
\]
\[
\text{Solve[}\{\text{D[uA0, x]} / \text{D[uA0, y]} = p_x / p_y, p_x * x + p_y * y = (m - T / 4)\}, \{x, y\}\}[1] \]

\[
\text{Out[114]} = \left\{ x \rightarrow \frac{p_y (4 m - T)}{4 p_x (p_x + p_y)}, y \rightarrow \frac{p_x (4 m - T)}{4 p_y (p_x + p_y)} \right\}
\]
General equilibrium for country A with public good when (proportional) taxation is adopted instead of Lindahl taxation, “GE with public good on commodity tax”, is defined by

\[
\begin{align*}
y_A^D &= y_A^S, \\
x_A^D &= x_A^S, \\
pz &= T \\
L_1^A + L_2^A + L_3^A &= L_{eA}, \\
K_1^A + K_2^A + K_3^A &= K_{eA}.
\end{align*}
\]

From the application of Newton method on (14), (15), (16), (17) and (18) we compute “the GE with public good on commodity tax” as in what follows.
\[
\text{In[119]} := \text{sol10} = \text{FindRoot}[\text{check400}, \{\text{px}, 4\}, \{\text{py}, 4\}, \{\text{pz}, 1\}, \\
\quad \{\text{wK}, 1\}, \{\text{T}, 1\}, \text{WorkingPrecision} \rightarrow 20, \\
\quad \text{AccuracyGoal} \rightarrow 20, \text{MaxIterations} \rightarrow 1000]
\]

\[
\text{Out[119]} = \{\text{px} \rightarrow 3.984296356837408302, \\
\quad \text{py} \rightarrow 4.1581708016145142109, \text{pz} \rightarrow 1.7891301135403672330, \\
\quad \text{wK} \rightarrow 0.79596080172148511115, \text{T} \rightarrow 0.069480915195488934423\}
\]

Note that we have exactly the same equilibrium prices for commodities and inputs.

\[
\text{In[120]} := \text{sol7}
\]

\[
\text{Out[120]} = \{\text{px} \rightarrow 3.984296356837408303, \\
\quad \text{py} \rightarrow 4.1581708016145142109, \text{wK} \rightarrow 0.79596080172148511116, \\
\quad \text{tL} \rightarrow 0.16930458013370953559, \text{tK} \rightarrow 0.3494502086076803549, \text{z} \rightarrow 0.038835026401741038141, \\
\quad \text{tL} \rightarrow 0.32656092211915612127\}
\]

\[
\text{In[121]} := \text{check400} \cdot . \text{sol10}
\]

\[
\text{Out[121]} = \{\text{True}, \text{True}, \text{True}, \text{True}, \text{True}\}
\]

Profit for the sector 3 is zero as shown in what follows.

\[
\text{In[122]} := (\text{pz} \cdot \text{z00} - (\text{wL} \cdot \text{demand}L3 + \text{wK} \cdot \text{demand}K3)) \cdot . \text{sol10} \cdot . \\
\quad \text{wL} \rightarrow 1 \cdot . \text{z} \rightarrow \text{z00}
\]

\[
\text{Out[122]} = 4.18 \cdot 10^{-18}
\]

On the one hand, workers’ income for “the GE with public good on poll tax” is greater than the one for “the GE with public good on income tax”.

\[
\text{In[123]} := \{\text{income}A$A$L0D} = \\
\quad (\text{wL} \cdot (2 / 3) \cdot \text{LeA} + \text{wK} \cdot (1 / 5) \cdot \text{KeA} - \text{T} / 4) \cdot . \text{wL} \rightarrow 1 \cdot . \text{sol10}, \\
\quad \text{income}A$A$L0B}
\]

\[
\text{Out[123]} = \{74.6089044550826455446, 74.6019945635721955316\}
\]

On the other hand, workers’ commodity consumptions for “the GE with public good on commodity tax” are not the same as those for “the GE with public good on income tax” as shown in what follows.
Thus, workers’ utility level for “the GE with public good on poll tax” is higher than the one for “the GE with public good on income tax” as shown in what follows..

On the one hand, capitalists’ income for “the GE with public good on poll tax” is greater than the one for “the GE with public good on income tax”.

On the other hand, capitalists’ consumptions of commodities for “the GE with public good on commodity tax” are not the same as those for “the GE with public good on income tax”.
Thus, capitalists’ utility level for “the GE with public good on commodity tax” is higher than the one for “the GE with public good on income tax” as shown in what follows.

\[
\text{In[130]} := \{uA0KD = uA /. \{x \rightarrow \text{demand}\$KD\$, y \rightarrow \text{demand}\$KYD\$, z \rightarrow z00\} /. \text{sol10} /. wL \rightarrow 1, uA1K\}
\]

\[
\text{Out[130]} = \{32.04192022433970819, 32.03751328193042107\}
\]

On the one hand, entrepreneur 1’s income for “the GE with public good on poll tax” is smaller than the one for “the GE with public good on income tax”.

\[
\text{In[131]} := \{\text{income}\$AE10D = \pi10 - T / 4 /. wL \rightarrow 1 /. \text{sol10}, \text{income}\$AE10B\}
\]

\[
\text{Out[131]} = \{17.50564710637009153, 17.51125389799448563\}
\]

On the other hand, entrepreneur 1’s consumptions of commodities for “the GE with public good on commodity tax” are not the same as those for “the GE with public good on income tax”.

\[
\text{In[132]} := \{x \rightarrow \text{demand}\$E1xD, y \rightarrow \text{demand}\$E1yD, z \rightarrow z00\} /. \text{sol10} /. wL \rightarrow 1
\]

\[
\text{Out[132]} = \{x \rightarrow 2.243294425257895280, y \rightarrow 2.060160923575372811, z \rightarrow 0.038835026401740385141\}
\]

\[
\text{In[133]} := \{x \rightarrow \text{demand}\$E1xB, y \rightarrow \text{demand}\$E1yB, z \rightarrow z00\} /. \text{sol18} /. wL \rightarrow 1
\]

\[
\text{Out[133]} = \{x \rightarrow 2.244613000925427384, y \rightarrow 2.061371856047919145, z \rightarrow 0.038835026401740385141\}
\]

Thus, entrepreneur 1’s utility level for “the GE with public good on poll tax” is lower than the one for “the GE with public good on income tax” as shown in what follows.

\[
\text{In[134]} := \{uA0E1D = uA /. \{x \rightarrow \text{demand}\$E1xD, y \rightarrow \text{demand}\$E1yD, z \rightarrow z00\} /. \text{sol10} /. wL \rightarrow 1, uA1E1\}
\]

\[
\text{Out[134]} = \{8.61457643907717261, 8.61963656091454375\}
\]

On the one hand, entrepreneur 2’s income for “the GE with public good on poll tax” is smaller than the one for “the GE with public good on income tax”.

\[
\text{In[135]} := \{\text{income}\$AE10D = \pi10 - T / 4 /. wL \rightarrow 1 /. \text{sol18}, \text{income}\$AE10B\}
\]

\[
\text{Out[135]} = \{17.50564710637009153, 17.51125389799448563\}
\]

On the other hand, entrepreneur 2’s consumptions of commodities for “the GE with public good on commodity tax” are not the same as those for “the GE with public good on income tax”.

\[
\text{In[136]} := \{x \rightarrow \text{demand}\$E1xB, y \rightarrow \text{demand}\$E1yB, z \rightarrow z00\} /. \text{sol18} /. wL \rightarrow 1
\]

\[
\text{Out[136]} = \{x \rightarrow 2.244613000925427384, y \rightarrow 2.061371856047919145, z \rightarrow 0.038835026401740385141\}
\]

Thus, entrepreneur 2’s utility level for “the GE with public good on poll tax” is lower than the one for “the GE with public good on income tax” as shown in what follows.
In[135]:= \{income$A$E20D = (pi20 - T / 4) /. wL -> 1 /. sol10,  
income$A$E20B\}

Out[135]= \{14.61084843064846863, 14.61747105347157406\}

On the other hand, entrepreneur 2’s consumptions of commodities for “the GE with public good on poll tax” are not the same as those for “the GE with public good on income tax”.

In[136]:= \{x \rightarrow \text{demand}$E2$x, y \rightarrow \text{demand}$E2$y, z \rightarrow z00\} /. sol10 /. wL -> 1

Out[136]= \{x \rightarrow 1.872334946180586074,  
y \rightarrow 1.719485078969246341, z \rightarrow 0.038835026401740385141\}

In[137]:= \{x \rightarrow \text{demand}$E2$x, y \rightarrow \text{demand}$E2$y, z \rightarrow z00\} /. sol8 /. wL -> 1

Out[137]= \{x \rightarrow 1.873803418403045445,  
y \rightarrow 1.720833671047042320, z \rightarrow 0.038835026401740385141\}

Thus, entrepreneur 2’s utility level for “the GE with public good on poll tax” is lower than the one for “the GE with public good on income tax” as shown in what follows.

In[138]:= \{uA0E2D =  
    uA / . \{x \rightarrow \text{demand}$E2$x, y \rightarrow \text{demand}$E2$y, z \rightarrow z00\} /.  
sol10 /. wL -> 1, uA1E2\}

Out[138]= \{7.190951456982733112, 7.196587174464033883\}

From the viewpoint of Bentham-type utilitarian, the income taxation (and commodity taxation) is more desirable than the Lindahl taxation and poll taxation, as shown in what follows. Furthermore, the poll taxation is the worst from the same viewpoint of Bentham-type utilitarian.

In[139]:= \{uA0L + uA0K + uA0E1 + uA0E2, uA1L + uA1K + uA1E1 + uA1E2,  
uA0LD + uA0KD + uA0E1D + uA0E2D\}

Out[139]= \{84.53727799264331946,  
    84.53727972251664468, 84.53727576387538624\}

Furthermore, the income taxation is most desirable taxation, and the poll taxation is the worst from the viewpoint of income distribution equality by the computation of Gini coefficients, as shown in what follows.
Conclusions

The aim of this paper is to combine the two traditions of public economics. The first tradition asks which taxation is desirable in order to impose a tax, income taxation or commodity taxation, without paying any attention on why the tax is necessary. The second one asserts that government imposes a tax in order to provide public goods. By constructing a primitive general equilibrium model which incorporates a public good, this paper asks which taxation is desirable in order to impose tax, income taxation, commodity taxation, or poll taxation to provide the public good. Formally, in Section I, we utilized the Lindahl mechanism to compute a Pareto-optimal public good level. The burden-sharing in this Lindahl mechanism may be regarded as a tax on the society members, while utilizing pseudo-market mechanism. We called it Lindahl taxation. In Section II, we computed the rate of income taxation in order to sustain the optimal public good level, and compared the Lindahl taxation and income taxation. We obtained the conclusion, in which the income taxation is more desirable than the Lindahl taxation from the utilitarian viewpoint, as well as from the viewpoint of Gini coefficient. In Section III, we computed the rate of (proportional) commodity taxation in order to sustain the optimal public good level, and compared the Lindahl taxation, income taxation and commodity taxation. We obtained exactly the same conclusion as in Section II, since we have exactly the same general equilibrium except for the tax rates. Finally, in Section IV, we examined if general equilibrium exists for the model with poll tax. It was shown that the general equilibrium exists for the model and the
poll taxation is the worst from the utilitarian viewpoint, as well as from the viewpoint of Gini coefficient.
This paper adopted the simulation approach in which production and utility functions are specified, and the parameters on these functions are also specified. Thus, the conclusions may depend on the specification. Further simulation with random selection of those parameters will be conducted in subsequent papers.

References


http://www.cc.aoyama.ac.jp/~fukito/IndexII.htm

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