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Introduction

There are two traditions in the argument on taxation. On the one hand, the argument on taxation is constructed in such a way that which taxation is desirable in order to impose a tax, income tax or commodity tax, without paying any attention on why the tax is necessary. On the other hand, it is also a tradition to assert that government imposes a tax in order to provide public goods. This paper combines the above two traditions by constructing a primitive general equilibrium model which incorporates a public good, asking which taxation is desirable in order to impose tax, income tax, commodity tax, or poll tax to provide the public good. Formally, first, we utilize the Lindahl mechanism to compute a Pareto-optimal public good level. The burden-sharing in this Lindahl mechanism may be regarded as a tax on the society members, while utilizing pseudo-market mechanism. We call it Lindahl tax. Next, we compute the rate of income tax in order to sustain the optimal public good level, and compare the Lindahl tax and income tax. We proceed to the computation of the rate of (proportional) commodity tax in order to sustain the optimal public good level, and compare the Lindahl tax, income tax and commodity tax. Finally we proceed to the comparison between the Lindahl tax, income tax, commodity tax and poll tax.

I: Economy with Public Good and Walras-Lindahl Mechanism

We start with constructing a primitive general equilibrium model which incorporates a public good. The production and utility functions are specified by particular functions, and their parameters are also specified. Utilizing Walras-Lindahl mechanism, the optimal level of public good, as well as burden-sharing of each member of the society, are derived.

- Production of country A

Country A is under national isolation. She has three sectors of production, which produces 3 goods, utilizing labor, $L_i$, and capital, $K_i$: $y$ stands for the output of sector 1, $x$ stands for the output of sector 2, and $z$ stands for the output of sector 3 ($i=1, 2, 3$), where $y$ and $x$ are private goods and $z$ is a public good. Production function of sector 1, $y = f_{1_A}$, is assumed as follows, with $a_1 + b_1 < 1$: decreasing returns to scale.
\[ f = L^a \cdot K^b; \quad c_1 = \{ a \to 1/6, \ b \to 1/5, \ L \to L1, \ K \to K1 \}; \]
\[ f_1 = f / . \ c1 \]
\[ Out[1] = K1^{1/5} \cdot L1^{1/6} \]

Production function of sector 2, \( x=f_2 A \), is assumed as follows, with \( a_2+b_2<1 \): decreasing returns to scale.

\[ c_2 = \{ a \to 1/4, \ b \to 1/3, \ L \to L2, \ K \to K2 \}; \quad f_2 = f / . \ c2 \]
\[ Out[2] = K2^{1/3} \cdot L2^{1/4} \]

Production function of sector 3, \( z=f_3 A \), is assumed as follows, with \( a_3+b_3=1 \: \): constant returns to scale.

\[ c_3 = \{ a \to 1/3, \ b \to 2/3, \ L \to L3, \ K \to K3 \}; \quad f_3 = f / . \ c3 \]
\[ Out[3] = K3^{2/3} \cdot L3^{1/3} \]

From the profit maximization of the sector 1, demand for labor, \( L_1 A^D \), and demand for capital, \( K_1 A^D \), are computed as in what follows, where \( p_y \) stands for the price of the consumption good, \( y \), \( w_L \), wage rate of labor, and \( w_K \), rental price of capital.

\[ \text{pi1} = p_y \ast f_1 - w_L \ast L1 - w_K \ast K1; \]
\[ \text{sol1} = \text{Solve}[[D[\text{pi1}, L1] == 0, D[\text{pi1}, K1] == 0], \{K1, L1\}]][[3]] \]
\[ Out[4] = \{ K1 \to p_y^{30/19} / (5 \times 6^{6/19} \cdot 6^{5/19} \cdot w_K^{25/19} \cdot w_L^{5/19}), \]
\[ L1 \to p_y^{30/19} / (6 \times 5^{6/19} \cdot 6^{5/19} \cdot w_K^{6/19} \cdot w_L^{24/19}) \} \]
\[ \text{In[5]} = \text{demand$\_L1} = L1 / . \ \text{sol1}; \ \text{demand$K1} = K1 / . \ \text{sol1}; \]

Thus, supply function of \( y \), \( y_A^S \), is computed as follows, with \( p_y \), \( w_L \), and \( w_K \), as parameters.

\[ \text{supply$A$} = \text{PowerExpand}[f_1 / . \ \text{sol1}]; \]

Profit function of sector 1, \( \pi_1 A \), is computed as follows, with \( p_y \), \( w_L \), and \( w_K \), as parameters. This profit accrues to entrepreneur 1.

\[ \text{pi1} = \text{PowerExpand}[\text{pi1} / . \ \text{sol1}]; \]

From the profit maximization of the sector 2, demand for labor, \( L_2 A^D \), and demand for capital, \( K_2 A^D \), as follows, where \( p_x \) stands for the price of the consumption good,
From the profit maximization of the sector 2, demand for labor, \( L_2 \), and demand for capital, \( K_2 \), as follows, where \( p_x \) stands for the price of the consumption good, \( x \).

\[
\text{In}[8] := \text{pi2} = p_x \cdot f_2 - w_L \cdot L_2 - w_K \cdot K_2;
\]

\[
\text{sol2} = \text{PowerExpand}[\text{Solve}[\{D[\text{pi2}, L_2] = 0, D[\text{pi2}, K_2] = 0\}, \{K_2, L_2\}]][[2]];
\]

\[
\text{In}[9] := \text{demand$L2} = \text{L2} /. \text{sol2}; \text{demand$K2} = \text{K2} /. \text{sol2};
\]

Thus, supply function of \( x, x^S \), is computed as follows, with \( p_x, w_L, \) and \( w_K, \) as parameters.

\[
\text{In}[10] := \text{supply$A$x} = \text{PowerExpand}[f_2 /. \text{sol2}];
\]

Profit function of sector 2, \( \pi_{2A} \), is computed as follows, with \( p_x, w_L, \) and \( w_K, \) as parameters. This profit accrues to entrepreneur 2.

\[
\text{In}[11] := \text{pi20} = \text{PowerExpand}[\text{pi2} /. \text{sol2}];
\]

The sector 3 produces a public good, \( z \), under constant returns to scale, so that demand for capital, \( K_{3A}^D \), and the one for labor, \( L_{3A}^D \), is derived by the minimizing cost, given output level \( z \), as in what follows.

\[
\text{In}[12] := \text{sol3} = \text{Solve}[\{f_3 = z, D[f_3, K_3]/D[f_3, L_3] = w_K/w_L\}, \{K_3, L_3\}][[2]]
\]

\[
\text{Out}[12] = \left\{ K_3 \rightarrow \frac{2^{1/3} w_L^{1/3} z}{w K^{1/3}}, L_3 \rightarrow \frac{w K^{2/3} z}{2^{2/3} w_L^{2/3}} \right\}
\]

\[
\text{In}[13] := \text{demand$L3} = \text{L3} /. \text{sol3}; \text{demand$K3} = \text{K3} /. \text{sol3};
\]

The homogeneity of degree one requires profit to be zero at the general equilibrium, so that \( p_z = w_L L_{3A}^D + w_K K_{3A}^D \) must hold at GE and \( p_z \) must be equal to the following when \( w_L = 1 \).

\[
\text{In}[14] := \text{hom} = \text{Simplify}[(w_L \cdot \text{demand$L3} + w_K \cdot \text{demand$K3}) / z] /. w_L \rightarrow 1
\]

\[
\text{Out}[14] = \frac{3 w K^{2/3}}{2^{2/3}}
\]

- Consumption of country A

We proceed to the demand side of country A. She is endowed with \( L_{cA} = 100 \) and \( K_{cA} = 50 \).
\[ \text{In[15] := } \text{LeA} = 100; \text{KeA} = 50; \]

All the agents in this paper: workers, capitalists, and 2 entrepreneurs, have the same CES utility function, \( u[y, x, z] = (y^a + x^a + z^a) ^ {1/a} \) which is specified as \( u[y, x, z] = (y^{1/2} + x^{1/2} + z^{1/2})^2 \): \( i.e. \ a = 1/2 \).

\[ \text{In[16] := } \text{uA} = (x^{(1/2)} + y^{(1/2)} + z^{(1/2)})^2 \]

\[ \text{Out[16] := } \left( \sqrt{x} + \sqrt{y} + \sqrt{z} \right)^2 \]

All the consumers maximize utility subject to income constraint:

\[
\begin{align*}
\text{max } u[y, x, z] & \quad \text{s.t. } p_x y + p_x x + \theta_j p_z z = m_j \quad (j=\text{L, K, 1, 2}) \\
\end{align*}
\]

where \( m_j \) is income and \( \theta_j \) is the burden share of the household \( j \) \( (j=\text{L, K, 1, 2, 3}) \). Worker (household \( \text{L} \))’s income, \( m_\text{L} \), consists of initial endowment of labor, evaluated by the wage rate: \( w_\text{L} L_{e\text{A}} \). It is assumed that they supply \( L_{e\text{A}} \) for labor supply. Capitalist (household \( \text{K} \))’s income, \( m_\text{K} \), consists of initial endowment of capital, evaluated by the rental price of capital: \( w_\text{K} K_{e\text{A}} \). It is assumed that they supply \( K_{e\text{A}} \) for capital supply. Entrepreneur 1 (household 1)’s income, \( m_1 \), consists of profit for the sector 1, \( \pi_{1A} \). Finally, entrepreneur 2 (household 2)’s income, \( m_2 \), consists of profit for the sector 2, \( \pi_{2A} \).

Demand function of workers for commodity \( y \), \( y_\text{L}^D \), that for commodity \( x \), \( x_\text{L}^D \), that for commodity \( z \), \( z_\text{L}^D \), demand function of capitalists for commodity \( y \), \( y_\text{K}^D \), that for commodity \( x \), \( x_\text{K}^D \), that for commodity \( z \), \( z_\text{K}^D \), demand function of entrepreneur 1 for commodity \( y \), \( y_{\text{E1}}^D \), that for commodity \( x \), \( x_{\text{E1}}^D \), that for commodity \( z \), \( z_{\text{E1}}^D \), and demand function of entrepreneur 2 for commodity \( y \), \( y_{\text{E2}}^D \), that for commodity \( x \), \( x_{\text{E2}}^D \), that for commodity \( z \), \( z_{\text{E2}}^D \), are derived as in what follows.

\[ \text{In[17] := } \text{sol13A =}\]

\[
\text{Solve}[[\text{D[uA, x]} / \text{D[uA, y]} == \text{px} / \text{py}, \]

\[
\text{D[uA, z]} / \text{D[uA, y]} == \text{tj} * \text{pz} / \text{py}, \]

\[
\text{px} * \text{x} + \text{py} * \text{y} + \text{tj} * \text{pz} * \text{z} == \text{m}], \{\text{x, y, z}\}]][[1]]
\]

\[ \text{Out[17] := } \{\text{x} \rightarrow (\text{mpy pz tj}) / (\text{px (px py + px pz tj + py pz tj)}), \]

\[
\text{y} \rightarrow (\text{mpy pz tj}) / (\text{py (px py + px pz tj + py pz tj)}), \]

\[
\text{z} \rightarrow (\text{mpy py}) / (\text{pz tj (px py + px pz tj + py pz tj)}) \} \]
\begin{verbatim}
In[18]:= demand$L$x = x /. sol3A /. \{m \rightarrow wL * LeA, t j \rightarrow tL\};
  demand$L$y = y /. sol3A /. \{m \rightarrow wL * LeA, t j \rightarrow tL\};
  demand$L$z = z /. sol3A /. \{m \rightarrow wL * LeA, t j \rightarrow tL\};
  demand$K$x = x /. sol3A /. \{m \rightarrow wK * KeA, t j \rightarrow tK\};
  demand$K$y = y /. sol3A /. \{m \rightarrow wK * KeA, t j \rightarrow tK\};
  demand$K$z = z /. sol3A /. \{m \rightarrow wK * KeA, t j \rightarrow tK\};
  demand$E1$x = x /. sol3A /. \{m \rightarrow pi10, t j \rightarrow t1\};
  demand$E1$y = y /. sol3A /. \{m \rightarrow pi10, t j \rightarrow t1\};
  demand$E1$z = z /. sol3A /. \{m \rightarrow pi10, t j \rightarrow t1\};
  demand$E2$x = x /. sol3A /. \{m \rightarrow pi20, t j \rightarrow t2\};
  demand$E2$y = y /. sol3A /. \{m \rightarrow pi20, t j \rightarrow t2\};
  demand$E2$z = z /. sol3A /. \{m \rightarrow pi20, t j \rightarrow t2\};
\{demand$L$y, demand$L$x, demand$L$z, demand$K$y,
  demand$K$x, demand$K$z, demand$E1$x, demand$E1$y,
  demand$E1$z, demand$E2$y, demand$E2$x, demand$E2$z\};

Country A’s demand for commodity $y$, $y_A^D$, is the sum of $y_L^D$, $y_K^D$, $y_{E1}^D$, $y_{E2}^D$.

In[19]:= demand$A$y =
  Simplify\[demand$L$y + demand$K$y + demand$E1$y +
  demand$E2$y\];

Country A’s demand for commodity $x$, $x_A^D$, is the sum of $x_L^D$, $x_K^D$, $x_{E1}^D$, $x_{E2}^D$.

In[20]:= demand$A$x =
  Simplify\[demand$L$x + demand$K$x + demand$E1$x +
  demand$E2$x\];

Country A’s consumption of commodity $z$ is determined by the Lindahl mechanism:
$z_A^D = z_L^D = z_K^D = z_{E1}^D = z_{E2}^D$ for suitable selection of $\theta_j$.

In[21]:= demand$L$z == demand$K$z == demand$E1$z == demand$E2$z;

■ “GE with Public good”

General equilibrium for country A with public good, “GE with public good”, is defined by

$$y_A^D = y_A^S,$$

$$x_A^D = x_A^S,$$

$$z_{AL}^D = z_{AK}^D = z_{AE1}^D = z_{AE2}^D = z,$$

$$L_1^D A + L_2^D A + L_3^D A = L_e A,$$
\[ K_{1A} D + K_{2A} D + K_{3A} D = K_{cA}. \]  

(6)

From the application of Newton method on (2), (3), (4) and (6) we compute the GE with public good as in what follows.

\[
\begin{align*}
\text{In[22]} & := \text{check100} = \\
& \{\text{supply}$A$x \rightarrow \text{demand}$A$x, \text{supply}$A$y \rightarrow \text{demand}$A$y, \\
& \text{demand}$K1 + \text{demand}$K2 + \text{demand}$K3 \rightarrow \text{KeA}, \\
& \text{demand}$L$z \rightarrow \\
& \text{(demand}$L$z + \text{demand}$K$z + \text{demand}$E1$z + \text{demand}$E2$z) / 4, \text{demand}$K$z \rightarrow \\
& \text{(demand}$L$z + \text{demand}$K$z + \text{demand}$E1$z + \text{demand}$E2$z) / 4, \text{demand}$E1$z \rightarrow \\
& \text{(demand}$L$z + \text{demand}$K$z + \text{demand}$E1$z + \text{demand}$E2$z) / 4, \text{z} \rightarrow \text{demand}$E2$z \}
\end{align*}
\]

\[ \{wL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow \text{hom}\}; \]

\[
\begin{align*}
\text{In[23]} & := \text{sol7} = \text{FindRoot} [\text{check100}, \{\text{px}, 10\}, \{\text{py}, 10\}, \{\text{wK}, 0.25\}, \\
& \{t1, 0.25\}, \{\text{z}, 10\}, \{tL, 0.25\}, \{tK, 0.25\}, \\
& \text{AccuracyGoal} \rightarrow 30, \text{WorkingPrecision} \rightarrow 20, \\
& \text{MaxIterations} \rightarrow 1000]
\end{align*}
\]

\[ \{\text{px} \rightarrow 14.153545624140019458, \\
\text{py} \rightarrow 19.279639252690992305, \text{wK} \rightarrow 3.8103703458622201692, \\
\text{t1} \rightarrow 0.06193278723926238167, \text{z} \rightarrow 56.38555537178640578, \\
\text{tL} \rightarrow 0.32500153599364843437, \text{tK} \rightarrow 0.55737371673138213739\} \]

It is confirmed that (5) is satisfied.

\[
\begin{align*}
\text{In[24]} & := \text{demand}$L1 + \text{demand}$L2 + \text{demand}$L3 - \text{LeA} / . \{wL \rightarrow 1\} / . \\
& \text{sol7}
\end{align*}
\]

\[ 0 \times 10^{-18} \]

Finally, it is confirmed that (4) is satisfied.

\[
\begin{align*}
\text{In[25]} & := \{\text{demand}$L$z \rightarrow \text{demand}$K$z, \text{demand}$K$z \rightarrow \text{demand}$E1$z, \\
& \text{demand}$E1$z \rightarrow \text{demand}$E2$z, \\
& \text{demand}$E2$z \rightarrow \text{demand}$L$z\} / . \\
& \{wL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow \text{hom}\} / . \text{sol7}
\end{align*}
\]

\[ \{\text{True, True, True, True}\} \]

This “GE with public good” is also derived by the following Walras-Lindahl differential equations.$\theta_j$
\[ \frac{dp_y(t)}{dt} = y_A D - y_A S \]
\[ \frac{dp_x(t)}{dt} = x_A D - x_A S \]
\[ \frac{dw_y(t)}{dt} = K_1 A D + K_2 A D + K_3 A D - K_{eA} \]
\[ \frac{d\theta_L(t)}{dt} = z_L D - (z_L D + z_K D + z_{E1} D + z_{E2} D)/4 \]
\[ \frac{d\theta_K(t)}{dt} = z_K D - (z_L D + z_K D + z_{E1} D + z_{E2} D)/4 \]
\[ \frac{d\theta_1(t)}{dt} = z_{E1} D - (z_L D + z_K D + z_{E1} D + z_{E2} D)/4 \]
\[ \frac{dz(t)}{dt} = z_{E2} D - z(t) \]

In[26]:= check101 =
{demand$A$x - supply$A$x, demand$A$y - supply$A$y,
 demand$K$1 + demand$K$2 + demand$K$3 - KeA,
 demand$L$z -
 (demand$L$z + demand$K$z + demand$E1$z +
 demand$E2$z) / 4,
 demand$K$z -
 (demand$L$z + demand$K$z + demand$E1$z +
 demand$E2$z) / 4,
 demand$E1$z -
 (demand$L$z + demand$K$z + demand$E1$z +
 demand$E2$z) / 4, demand$E2$z - z} /. 
{wl \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow hom} /. 
{py \rightarrow py[t], px \rightarrow px[t], wK \rightarrow wK[t], t1 \rightarrow t1[t], 
 tL \rightarrow tL[t], tK \rightarrow tK[t], z \rightarrow z[t]};

In[27]:= check102 = {D[px[t], t] \rightarrow check101[[1]], 
D[py[t], t] \rightarrow check101[[2]], 
D[wK[t], t] \rightarrow check101[[3]], 
D[tL[t], t] \rightarrow check101[[4]], 
D[tK[t], t] \rightarrow check101[[5]], 
D[t1[t], t] \rightarrow check101[[6]], 
D[z[t], t] \rightarrow check101[[7]], py[0] \rightarrow 1, px[0] \rightarrow 1, 
wK[0] \rightarrow 1, t1[0] \rightarrow 0.1, tL[0] \rightarrow 0.1, tK[0] \rightarrow 0.1, 
z[0] \rightarrow 1};
\begin{verbatim}
In[28]:= sol101 = NDSolve[check102, {py, px, wK, tL, tK, z}, {t, 0, 1000}]

Out[28]= {{py \to InterpolatingFunction[{{0., 1000.}}, <>],
        px \to InterpolatingFunction[{{0., 1000.}}, <>],
        wK \to InterpolatingFunction[{{0., 1000.}}, <>],
        t1 \to InterpolatingFunction[{{0., 1000.}}, <>],
        tL \to InterpolatingFunction[{{0., 1000.}}, <>],
        tK \to InterpolatingFunction[{{0., 1000.}}, <>],
        z \to InterpolatingFunction[{{0., 1000.}}, <>]}}

Starting from the initial values: \( p_y(0)=1, \ p_x(0)=1, \ w_K(0)=1, \ \theta_L(0)=0.1, \ \theta_K(0)=0.1, \ \theta_1(0)=0.1, \) and \( z(0)=1; \) the trajectories on the differential equations: \( p_y(t), \ p_x(t), \)
\( w_K(t), \ \theta_L(t), \ \theta_K(t), \ \theta_1(t), \) and \( z(t); \) converge to the “GE with public good” as \( t \)
approaches 1000.

In[29]:= {py[1000], px[1000], wK[1000], t1[1000], tL[1000],
        tK[1000], z[1000]} /. sol101

Out[29]= {19.2796, 14.1535, 3.81037,
           0.0619328, 0.325002, 0.557374, 56.3856}

The “GE with public good” income for the (aggregate) workers before and after the
 deduction of the burden of consuming public good, are computed as follows.

In[30]:= {income$A$L0A =
        wL \* LeA /. {wL \to 1, t2 \to 1 - t1 - tL - tK, pz \to hom} /. sol7,
        income$A$L0B =
        (wL \* LeA - tj \* pz \* z) /. {tj \to tL, z \to demand$L$z} /. sol7
           {wL \to 1, t2 \to 1 - t1 - tL - tK, pz \to hom} /. sol7}

Out[30]= {100, 15.5111639395550368}

Its utility level at the GE with public good is computed as in what follows.

In[31]:= uA0L =
        uA /. {x \to demand$L$x, y \to demand$L$y, z \to demand$L$z} /. sol7
           {wL \to 1, t2 \to 1 - t1 - tL - tK, pz \to hom} /. sol7

Out[31]= 78.98947171601549650

The workers’ consumption of commodities are computed as in what follows.
\end{verbatim}
In[32]:= \{L$x0A =
  demand$L$x /. \{wL \to 1, t2 \to 1 - t1 - tL - tK, pz \to hom\} /.
sol7,
L$y0A =
  demand$L$y /. \{wL \to 1, t2 \to 1 - t1 - tL - tK, pz \to hom\} /.
sol7\}

Out[32]= \{0.6319755730104923171, 0.3405908558529144362\}

The “GE with public good” income for the (aggregate) capitalists before and after the deduction of the burden of consuming public good, are computed as follows.

In[33]:= \{income$A$K0A =
  (wK*KeA) /. \{wL \to 1, t2 \to 1 - t1 - tL - tK, pz \to hom\} /.
sol7,
income$A$K0B =
  (wK*KeA - tj*pz*z) /. \{tj \to tK, z \to demand$K$z\} /.
  \{wL \to 1, t2 \to 1 - t1 - tL - tK, pz \to hom\} /. sol7\}

Out[33]= \{190.51851729311100846, 45.62118185947570415\}

Its utility level at the GE with public good is computed as in what follows.

In[34]:= uA0K =
  uA /. \{x \to demand$K$x, y \to demand$K$y, z \to demand$K$z\} /.
  \{wL \to 1, t2 \to 1 - t1 - tL - tK, pz \to hom\} /. sol7

Out[34]= 97.48133120570986636

The capitalists’ consumption of commodities are computed as in what follows.

In[35]:= \{K$x0A =
  demand$K$x /. \{wL \to 1, t2 \to 1 - t1 - tL - tK, pz \to hom\} /.
sol7,
K$y0A =
  demand$K$y /. \{wL \to 1, t2 \to 1 - t1 - tL - tK, pz \to hom\} /.
sol7\}

Out[35]= \{1.858756226122657164, 1.0017402585060650446\}

The “GE with public good” income for the entrepreneur 1 before and after the deduction of the burden of consuming public good, are computed as follows.
In[36]:= \{income\$A\$E10A = \
    (pi10) / . \{tj \rightarrow t1, z \rightarrow demand\$E1\$z\} /.
    \{WL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow hom\} / . sol7, \
    income\$A\$E10B = \
    (pi10 - tj \* pz \* z) / . \{tj \rightarrow t1, z \rightarrow demand\$E1\$z\} /.
    \{WL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow hom\} / . sol7\}

Out[36]= \{16.663588816245526200, 0.56326765368126488\}

Its utility level at the GE with public good is computed as in what follows.

In[37]:= uA0E1 = \
    uA / . \{x \rightarrow demand\$E1\$x, y \rightarrow demand\$E1\$y, \
    z \rightarrow demand\$E1\$z\} /.
    \{WL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow hom\} / . sol7

Out[37]= 60.39985087702961566

The entrepreneur 1’s consumption of commodities are computed as in what follows.

In[38]:= \{E1\$x0A = \
    demand\$E1\$x / . \{WL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow hom\} / .
    sol7, \
    E1\$y0A = \
    demand\$E1\$y / . \{WL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow hom\} / .
    sol7\}

Out[38]= \{0.02294936728027992340, 0.01236811196046597828\}

The “GE with public good” income for the entrepreneur 2 before and after the deduction of the burden of consuming public good, are computed as follows.

In[39]:= \{income\$A\$E20A = \
    (pi20) / . \{tj \rightarrow t2, z \rightarrow demand\$E2\$z\} / . sol7 / . WL \rightarrow 1, \
    income\$A\$E20B = \
    (pi20 - tj \* pz \* z) / . \{tj \rightarrow t2, z \rightarrow demand\$E2\$z\} /.
    \{WL \rightarrow 1, t2 \rightarrow 1 - t1 - tL - tK, pz \rightarrow hom\} / . sol7\}

Out[39]= \{14.933396863935359797, 0.45546873059374796\}

Its utility level at the GE with public good is computed as in what follows.
The entrepreneur 2’s consumption of commodities are computed as in what follows.

\[
\text{In}[41]:= \text{\{E2$\times0A = } \text{demand$E2$x} / . \text{\{wL} \rightarrow 1, \text{t2} \rightarrow 1 - \text{t1} - \text{tL} - \text{tK}, \text{pz} \rightarrow \text{hom} \} / . \text{sol7}
\]

\text{Out}[41]= \{0.01855728642460560998, 0.01000108601596087427\}

Using the “modified GE with public good” incomes for 4 agents, the Gini coefficient before the deduction of the burden of consuming public good, \(gini_l\), is computed.

\[
\text{In}[42]:= \text{giniA[y\_]} := \text{Module[\{z1, z2\}, z1 = Sort[y];}
\]
\[
\text{z2 = (y[[1]] + y[[2]] + y[[3]] + y[[4]]) / 4;}
\]
\[
1 + (1/4) - 2(1*z1[[4]] + 2*z1[[3]] + 3*z1[[2]] + 4*z1[[1]]) / (4^2*z2);}
\]
\[
\text{giniA[\{income$A$LOA, income$A$KOA, income$A$E10A, income$A$E20A\]}
\]

\text{Out}[43]= 0.4735038882324976043

Meanwhile the Gini coefficient after the deduction of the burden of consuming public good is computed as in what follows.

\[
\text{In}[44]:= \text{giniA[\{income$A$LOB, income$A$KOB, income$A$E10B, income$A$E20B\]}
\]

\text{Out}[44]= 0.605158552303251932

In this specified case, the income distribution inequality of country A is more unequal when the burden of consuming public good is deducted from each member’s income.

The optimal public good level is computed as in what follows.
\textbf{II : Income Tax Provision of Public Good}

In this section, we examine if it is possible to use income taxation instead to Lindahl taxation in order to achieve the optimum public good level, \( z_0 = 56.38555537178640578 \). Since \( z_0 \) is provided for each member of the society, the utility function becomes the following one \( u[y, x, z_0] \).

\[
\text{In}[48] := \text{uA0} = \text{uA} / . \ z \rightarrow \text{z00}
\]

\[
\text{Out}[48] = \left( 7.5090315978279543113 + \sqrt{x} + \sqrt{y} \right)^2
\]

All the consumers maximize utility subject to income constraint:

\[
\text{max} \ u[y, x, z_0] \ \text{s.t.} \ p_y y + p_x x = (1 - \tau_y) m_j \ (j = L, K, 1, 2) \quad (7)
\]

where \( m_j \) is income and \( \tau_y \) is the rate of income tax \( (j = L, K, 1, 2, 3) \). Worker (household \( L \)’s income, \( m_L \), consists of initial endowment of labor, evaluated by the wage rate: \( w_L L_{cA} \). It is assumed that they supply \( L_{cA} \) for labor supply. Capitalist (household \( K \)’s income, \( m_L \), consists of initial endowment of capital, evaluated by the rental price of capital: \( w_K K_{cA} \). It is assumed that they supply \( K_{cA} \) for capital supply. Entrepreneur 1 (household 1)’s income, \( m_1 \), consists of profit for the sector 1, \( \pi_{1A} \). Finally, entrepreneur 2 (household 2)’s income, \( m_2 \), consists of profit for the sector 2, \( \pi_{2A} \).

Demand function of workers for commodity \( y \), \( y_L^D \), that for commodity \( x \), \( x_L^D \),
demand function of capitalists for commodity \( y \), \( y_K^D \), that for commodity \( x \), \( x_K^D \), demand function of entrepreneur 1 for commodity \( y \), \( y_{E1}^D \), that for commodity \( x \), \( x_{E1}^D \), and demand function of entrepreneur 2 for commodity \( y \), \( y_{E2}^D \), and that for commodity \( x \), \( x_{E2}^D \), are derived as in what follows.

\[
sol3B = \text{Solve}\left[\left\{\frac{D[uA0, x]}{D[uA0, y]} = \frac{px}{py}, \right.\nonumber \\
px \cdot x + py \cdot y = (1 - ty) \cdot m, \left.\right\}, \{x, y\}\right]\nonumber
\]

\[
Out[49]= \left\{\frac{x \rightarrow \frac{m \cdot py - m \cdot px \cdot ty}{px \cdot (px + py)}, \ y \rightarrow \frac{m \cdot px - m \cdot px \cdot ty}{py \cdot (px + py)}\right\}
\]

\[
In[50]= \text{demand$L$yB} = x /\text{. sol3B /\{m \rightarrow wL \* LeA\};}\nonumber \\
demand$L$yB = y /\text{. sol3B /\{m \rightarrow wL \* LeA\};}\nonumber \\
demand$K$xB = x /\text{. sol3B /\{m \rightarrow wK \* KeA\};}\nonumber \\
demand$K$yB = y /\text{. sol3B /\{m \rightarrow wK \* KeA\};}\nonumber \\
demand$E1$xB = x /\text{. sol3B /\{m \rightarrow pi10\};}\nonumber \\
demand$E1$yB = y /\text{. sol3B /\{m \rightarrow pi10\};}\nonumber \\
demand$E2$xB = x /\text{. sol3B /\{m \rightarrow pi20\};}\nonumber \\
demand$E2$yB = y /\text{. sol3B /\{m \rightarrow pi20\};}\nonumber
\]

\[
In[51]= \text{demand$A$yB} = \text{Simplify}\left[\text{demand$L$yB} + \text{demand$K$yB} + \text{demand$E1$yB} + \text{demand$E2$yB}\right];
\]

\[
In[52]= \text{demand$A$xB} = \text{Simplify}\left[\text{demand$L$xB} + \text{demand$K$xB} + \text{demand$E1$xB} + \text{demand$E2$xB}\right];
\]

General equilibrium for country A with public good when income taxation is adopted instead of Lindahl taxation, “GE with public good on income tax”, is defined by

\[
y_A^D = y_A^S, \tag{8}
\]
\[
x_A^D = x_A^S, \tag{9}
\]
\[
p;z = \tau_y (w_L L_{cA} + w_K K_{eA} + \pi_{1A} + \pi_{2A}) \tag{10}
\]
\[
L_1 A^D + L_2 A^D + L_3 A^D = L_{cA} \tag{11}
\]
\[
K_1 A^D + K_2 A^D + K_3 A^D = K_{eA} \tag{12}
\]

From the application of Newton method on (8), (9), (10) and (12) we compute the GE with public good on income tax as in what follows.
In[53]:=  
\[
\text{check200 = } \\
\{\text{supply}A\times \text{demand}A\times B, \text{supply}A\times y \text{demand}A\times y B, \\
\text{demand}K1 + \text{demand}K2 + \text{demand}K3 \text{ Ke}A, \\
\text{demand}L1 + \text{demand}L2 + \text{demand}L3 \text{ Le}A, \\
pz \times z00 = ty (wL*LeA + wK*KeA + pi10 + pi20) \}/. \\
\{wL \rightarrow 1, z \rightarrow z00\}; \\
\]

In[54]:=  
\[
\text{sol8 = FindRoot[check200, \{px, 1\}, \{py, 1\}, \{pz, 1\}, \\
\{wK, 1\}, \{ty, 0.3\}, \text{AccuracyGoal }\rightarrow 30, \\
\text{WorkingPrecision }\rightarrow 20, \text{MaxIterations }\rightarrow 1000] \\
\]

Out[54]=  
\[
\{px \rightarrow 14.153545624140019458, \\
py \rightarrow 19.279639252690992305, pz \rightarrow 4.61047901979988818, \\
wK \rightarrow 3.810370345862201692, ty \rightarrow 0.80705342770025117950\} \\
\]

Note that we have exactly the same equilibrium prices for commodities and inputs.

In[55]:=  
\[
\text{sol7} \\
\]

Out[55]=  
\[
\{px \rightarrow 14.153545624140019458, \\
py \rightarrow 19.279639252690992305, wK \rightarrow 3.810370345862201692, \\
t1 \rightarrow 0.061932787239262383167, z \rightarrow 56.38555537178640578, \\
tL \rightarrow 0.32500153599364843437, tK \rightarrow 0.55737371673138213739\} \\
\]

In[56]:=  
\[
\text{check200 /. sol8} \\
\]

Out[56]=  
\[
\{\text{True, True, True, True, True}\} \\
\]

Profit for the sector 3 is zero as shown in what follows.

In[57]:=  
\[
(pz \times z00 == wL*\text{demand}L3 + wK*\text{demand}K3) \/. \text{sol8} /. \\
\{wL \rightarrow 1 \}/. z \rightarrow z00 \\
\]

Out[57]=  
\[
\text{True} \\
\]

Furthermore, the price of public good in this section is exactly the same one as in the previous section.

In[58]:=  
\[
pzO \\
\]

Out[58]=  
\[
4.6104790191979988818 \\
\]

The “GE with public good on income tax” income for the (aggregate) workers is computed as follows.
\begin{align*}
\text{In[59]} &:= \text{incomeA$L1B} = (1 - \text{ty}) \ wL * \text{LeA} / \ wL \rightarrow 1 / \ . \ \text{sol8} \\
\text{Out[59]} &= 19.294657229974882050 \\
&\text{It is greater than the income in “GE with public good” after the deduction of the burden of consuming public good.} \\
\text{In[60]} &:= \text{incomeA$L0B} \\
\text{Out[60]} &= 15.511639395550368 \\
&\text{The “GE with public good on income tax” utility level for the (aggregate) workers is computed as follows.} \\
\text{In[61]} &:= \text{uA1L} = \\
&\quad \text{uA} / \ . \ \{x \to \text{demand$L$xB, y \to \text{demand$L$yB, z \to z00} / \ . \ \text{sol8} / \ . \ wL \rightarrow 1} \\
\text{Out[61]} &= 81.84038782010972444 \\
&\text{It is higher than the utility in “GE with public good”.} \\
\text{In[62]} &:= \text{uA0L} \\
\text{Out[62]} &= 78.98947171601549650 \\
&\text{The workers’ consumption of commodities are computed as in what follows.} \\
\text{In[63]} &:= \{L$x1A = \text{demand$L$xB} / \ . \ \{z \to z00} / \ . \ \text{sol8} / \ . \ wL \rightarrow 1, \\
&\quad \text{L$y1A = \text{demand$L$yB} / \ . \ \{z \to z00} / \ . \ \text{sol8} / \ . \ wL \rightarrow 1} \\
\text{Out[63]} &= \{0.786127469638738490, 0.4236680009929730177\} \\
&\text{Compare this with the consumptions in the previous section.} \\
\text{In[64]} &:= \{L$x0A, L$y0A} \\
\text{Out[64]} &= \{0.6319755730104923171, 0.340590855852914362\} \\
&\text{The “GE with public good on income tax” income for the (aggregate) capitalists is computed as follows.} \\
\text{In[65]} &:= \text{incomeA$K1B} = (1 - \text{ty}) * \text{wK} * \text{KeA} / \ wL \rightarrow 1 / \ . \ \text{sol8} \\
\text{Out[65]} &= 36.75989487133618914 \\
&\text{It is smaller than the income in “GE with public good” after the deduction of the}
burden of consuming public good.

\[\text{In}[66] = \text{income}\{\text{A}, \text{K}0\B\}\]

\[\text{Out}[66] = 45.62118185947570415\]

The “GE with public good on income tax” utility level for the (aggregate) capitalists is computed as follows.

\[\text{In}[67] = \text{uA1K} = \text{uA} / \{\text{x} \rightarrow \text{demand}\{\text{K}, \text{x}\}, \text{y} \rightarrow \text{demand}\{\text{K}, \text{y}\}, \text{z} \rightarrow \text{z00}\} / \text{sol8} / \text{wL} \rightarrow 1\]

\[\text{Out}[67] = 92.76133922986662403\]

It is lower than the utility in “GE with public good”.

\[\text{In}[68] = \text{uA0K}\]

\[\text{Out}[68] = 97.48133120570986636\]

The capitalists’ consumption of commodities are computed as in what follows.

\[\text{In}[69] = \{\text{Kx1A} = \text{demand}\{\text{K}, \text{x}\} / \{\text{z} \rightarrow \text{z00}\} / \text{sol8} / \text{wL} \rightarrow 1, \text{Ky1A} = \text{demand}\{\text{K}, \text{y}\} / \{\text{z} \rightarrow \text{z00}\} / \text{sol8} / \text{wL} \rightarrow 1\}\]

\[\text{Out}[69] = \{1.497718399189575982, 0.807165993737175018\}\]

Compare this with the consumptions in the previous section.

\[\text{In}[70] = \{\text{Kx0A}, \text{Ky0A}\}\]

\[\text{Out}[70] = \{1.858756226122657164, 1.0017402585060650446\}\]

The “GE with public good on income tax” income for the entrepreneur 1 is computed as follows.

\[\text{In}[71] = \text{income}\{\text{A}, \text{E11B}\} = (1 - \text{ty}) * \text{pi10} / \text{wL} \rightarrow 1 / \text{sol8}\]

\[\text{Out}[71] = 3.215182344307003283\]

It is greater than the income in “GE with public good” after the deduction of the burden of consuming public good.

\[\text{In}[72] = \text{income}\{\text{A}, \text{E10B}\}\]

\[\text{Out}[72] = 0.56326765368126488\]
The “GE with public good on income tax” utility level for the entrepreneur 1 is computed as follows.

\[ u_{A1} = \frac{u_{A}}{\{x \rightarrow \text{demand}\_E1\_xB, y \rightarrow \text{demand}\_E1\_yB, z \rightarrow z00\} / . \text{sol8} / . wL \rightarrow 1} \]

\[ \text{Out[73]}= 66.20540091536471678 \]

It is higher than the utility in “GE with public good”.

\[ u_{A0} \]

\[ \text{Out[74]}= 60.39985087702961566 \]

The entrepreneur 1’s consumption of commodities are computed as in what follows.

\[ \{E1\_x1A = \text{demand}\_E1\_xB / . \{z \rightarrow z00\} / . \text{sol8} / . wL \rightarrow 1, \]
\[ \quad E1\_y1A = \text{demand}\_E1\_yB / . \{z \rightarrow z00\} / . \text{sol8} / . wL \rightarrow 1\} \]

\[ \text{Out[75]}= \{0.1309970491121547715, 0.0705982936314760367\} \]

Compare this with the consumptions in the previous section.

\[ \{E1\_x0A, E1\_y0A\} \]

\[ \text{Out[76]}= \{0.02294936728027992340, 0.012368111960465978728\} \]

The “GE with public good on income tax” income for the entrepreneur 2 is computed as follows.

\[ \text{income}\_A\_E21B = (1 - ty) * \pi_{20} / . wL \rightarrow 1 / . \text{sol8} \]

\[ \text{Out[77]}= 2.881347737688146198 \]

It is greater than the income in “GE with public good” after the deduction of the burden of consuming public good.

\[ \text{income}\_A\_E20B \]

\[ \text{Out[78]}= 0.45546873059374796 \]

The “GE with public good on income tax” utility level for the entrepreneur 2 is computed as follows.
In[79]:= uA2 = uA / {x \to demandx, y \to demandy, z \to z0} / sol8 / wL \to 1

Out[79]= 65.66174123232441454

It is higher than the utility in “GE with public good”.

In[80]:= uA2

Out[80]= 59.9890859141016334

The entrepreneur 2’s consumption of commodities are computed as in what follows.

In[81]:= \{E2x, E2y\} = demandx / sol8 / wL \to 1,
   \{E2y\} = demandy / sol8 / wL \to 1

Out[81]= \{0.1173955348975657714, 0.0632680239737822616\}

Compare this with the consumptions in the previous section.

In[82]:= \{E2x0, E2y0\}

Out[82]= \{0.0185572864260560998, 0.01000108601596087427\}

From the viewpoint of Bentham-type utilitarian, the income taxation is more desirable than the Lindahl taxation, since the sum of utility for the former case is greater than the one for the latter case, as shown in what follows.

In[83]:= \{UA1L + UA0K + UA0E1 + UA0E2, UA1L + UA1K + UA1E1 + UA1E2\}

Out[83]= \{296.8597397128566119, 306.4688691976654798\}

Furthermore, the income taxation is more desirable than the Lindahl taxation in the sense that the Gini coefficient for the former is smaller than the latter case, as shown in what follows.

In[84]:= giniA[\{incomeA$L1B, incomeA$K1B, incomeA$E11B, incomeA$E21B\}]

Out[84]= 0.4735038882324976043

In[85]:= giniA[\{incomeA$L0B, incomeA$K0B, incomeA$E10B, incomeA$E20B\}]

Out[85]= 0.605158552303251932

Note that the Gini coefficient for the income taxation is exactly the same as in the one for the incomes before the Lindahl taxation.
Note that the Gini coefficient for the income taxation is exactly the same as in the one for the incomes before the Lindahl taxation.

\[
\text{In[86]} := \text{giniA[]}\{\text{incomeA$L0A, incomeA$K0A, incomeA$E10A, incomeA$E20A}}\}
\]

\[
\text{Out[86]} = 0.473503882324976043
\]

### III : (Proportional) Commodity Tax Provision of Public Good

In this section, we examine if it is possible to use (proportional) commodity taxation instead to Lindahl taxation in order to achieve the optimum public good level, \( z_O = 56.38555537178640578 \). Since \( z_O \) is provided for each member of the society, the utility function becomes the following one \( u[y, x, z_O] \).

\[
\text{In[87]} := uA0 = uA / . \; z \to z00
\]

\[
\text{Out[87]} = (7.5090315978279543113 + \sqrt{y} + \sqrt{x})^2
\]

All the consumers maximize utility subject to income constraint:

\[
\max u[y, x, z_O] \; \text{s.t.} \; (1 + \tau_1)p_ym_j + (1 + \tau_1)p_xm_j (j = L, K, 1, 2) \tag{13}
\]

where \( m_j \) is income and \( \tau_j \) is the rate of commodity tax \((j = L, K, 1, 2, 3)\). Worker (household \( L \))’s income, \( m_L \), consists of initial endowment of labor, evaluated by the wage rate: \( w_LLrA \). It is assumed that they supply \( LrA \) for labor supply. Capitalist (household \( K \))’s income, \( m_K \), consists of initial endowment of capital, evaluated by the rental price of capital: \( w_KKrA \). It is assumed that they supply \( KrA \) for capital supply. Entrepreneur 1 (household 1)’s income, \( m_1 \), consists of profit for the sector 1, \( \pi_{1A} \). Finally, entrepreneur 2 (household 2)’s income, \( m_2 \), consists of profit for the sector 2, \( \pi_{2A} \).

Demand function of workers for commodity \( y \), \( y_{Ld} \), that for commodity \( x \), \( x_{Ld} \), demand function of capitalists for commodity \( y \), \( y_{Kd} \), that for commodity \( x \), \( x_{Kd} \), demand function of entrepreneur 1 for commodity \( y \), \( y_{Ed} \), that for commodity \( x \), \( x_{Ed} \), and demand function of entrepreneur 2 for commodity \( y \), \( y_{E2d} \), and that for commodity \( x \), \( x_{E2d} \), are derived as in what follows.
In[88]:= \text{sol3C = Solve}\left\{D[uA0, x] / D[uA0, y] = px / py, \right. \\
\left. (1 + t1) * px * x + (1 + t1) * py * y = m\right\}, \{x, y\}\right\}[[1]]

Out[88]= \left\{x \to \frac{m \, py}{px \, (px + py) \, (1 + t1)}, \quad y \to \frac{m \, px}{py \, (px + py) \, (1 + t1)}\right\}

In[89]:= demand$L$xC = x /. sol3C /. \{m \to wL \ast LeA\};
   demand$L$yC = y /. sol3C /. \{m \to wL \ast LeA\};
   demand$K$xC = x /. sol3C /. \{m \to wK \ast KeA\};
   demand$K$yC = y /. sol3C /. \{m \to wK \ast KeA\};
   demand$E1$xC = x /. sol3C /. \{m \to pi10\};
   demand$E1$yC = y /. sol3C /. \{m \to pi10\};
   demand$E2$xC = x /. sol3C /. \{m \to pi20\};
   demand$E2$yC = y /. sol3C /. \{m \to pi20\};

In[90]:= demand$A$yC =
   Simplify[demand$L$yC + demand$K$yC + demand$E1$yC +
            demand$E2$yC];

In[91]:= demand$A$xC =
   Simplify[demand$L$xC + demand$K$xC + demand$E1$xC +
            demand$E2$xC];

General equilibrium for country A with public good when (proportional) taxation is
adopted instead of Lindahl taxation, “GE with public good on commodity tax”, is
defined by

\begin{align*}
   y_A^D &= y_A^S, \\
   x_A^D &= x_A^S, \\
   p_z &= \tau_1 p_y \left( y_L^D + y_K^D + y_{E1}^D + y_{E2}^D \right) + \\
   & \quad \tau_1 p_x \left( x_L^D + x_K^D + x_{E1}^D + x_{E2}^D \right) \\
   L_1^A + L_2^A + L_3^A &= L_{eA}, \\
   K_1^A + K_2^A + K_3^A &= K_{eA}.
\end{align*}

From the application of Newton method on (14) ~ (18) we compute “the GE with
public good on commodity tax” as in what follows.
In[92]:= check300 =  
   {supply$A$x == demand$A$xC, supply$A$y == demand$A$yC,  
    demand$A$K1 + demand$A$K2 + demand$A$K3 == KeA,  
    demand$L1 + demand$L2 + demand$L3 == LeA,  
    pz * z00 == t1 * px * demand$A$xC + t1 * py * demand$A$yC} /.  
   {wL -> 1, z -> z00};

In[93]:= sol9 = FindRoot[check300, {px, 15}, {py, 17},  
   {pz, 3}, {wK, 3}, {t1, 0.2}]
Out[93]= {px -> 14.1535, py -> 19.2796,  
   pz -> 4.61048, wK -> 3.81037, t1 -> 4.18278}

Note that we have exactly the same equilibrium prices for commodities and inputs.

In[94]:= sol7
Out[94]= {px -> 14.153545624140019458,  
   py -> 19.279639252690992305, wK -> 3.8103703458622201692,  
   t1 -> 0.06193278723926238167, z -> 56.38555537178640578,  
   tL -> 0.32500153599364843437, tK -> 0.55737371673138213739}

In[95]:= check300 /. sol9
Out[95]= {True, True, True, True, True, True}

Profit for the sector 3 is zero as shown in what follows.

In[96]:= (pz * z00 == wL * demand$A$L3 + wK * demand$A$K3) /. sol9 /.  
    wL -> 1 /. z -> z00
Out[96]= True

On the one hand, workers’ income for “the GE with public good on commodity tax” is  
greater than the one for “the GE with public good on income tax”.

In[97]:= {income$A$L0C = wL * LeA /. wL -> 1 /. sol9, income$A$L0B}
Out[97]= {100, 15.51116393955550368}

On the other hand, workers’ commodity consumptions for “the GE with public good on  
commodity tax” are exactly the same as those for “the GE with public good on  
income tax” as shown in what follows.
Thus, workers’ utility level for “the GE with public good on commodity tax” is exactly the same as the one for “the GE with public good on income tax” as shown in what follows.

On the one hand, capitalists’ income for “the GE with public good on commodity tax” is greater than the one for “the GE with public good on income tax”.

On the other hand, capitalists’ consumptions of commodities for “the GE with public good on commodity tax” are exactly the same as those for “the GE with public good on income tax”.

Thus, capitalists’ utility level for “the GE with public good on commodity tax” is exactly the same as the one for “the GE with public good on income tax” as shown in what follows.
Thus, capitalists’ utility level for “the GE with public good on commodity tax” is exactly the same as the one for “the GE with public good on income tax” as shown in what follows.

On the one hand, entrepreneur 1’s income for “the GE with public good on commodity tax” is greater than the one for “the GE with public good on income tax”.

On the other hand, entrepreneur 1’s consumptions of commodities for “the GE with public good on commodity tax” are exactly the same as those for “the GE with public good on income tax”.

Thus, entrepreneur 1’s utility level for “the GE with public good on commodity tax” is exactly the same as the one for “the GE with public good on income tax” as shown in what follows.

On the one hand, entrepreneur 2’s income for “the GE with public good on commodity tax” is greater than the one for “the GE with public good on income tax”.

On the other hand, entrepreneur 2’s consumptions of commodities for “the GE with public good on commodity tax” are exactly the same as those for “the GE with public good on income tax”.

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good on income tax”.

\[
\begin{align*}
\text{In[110]} &= \{ x \to \text{demand} \times C, y \to \text{demand} \times C, z \to z00 \} / . \text{sol9} / . \\
& \quad \text{wL} \to 1 \\
\text{Out[110]} &= \{ x \to 0.117396, y \to 0.063268, z \to 56.38555537178640578 \} \\
\text{In[111]} &= \{ x \to \text{demand} \times B, y \to \text{demand} \times B, z \to z00 \} / . \text{sol8} / . \\
& \quad \text{wL} \to 1 \\
\text{Out[111]} &= \{ x \to 0.1173955348975657714, \\
& \quad y \to 0.0632680239737822616, z \to 56.38555537178640578 \}
\end{align*}
\]

Thus, entrepreneur 2’s utility level for “the GE with public good on commodity tax” is
exactly the same as the one for “the GE with public good on income tax” as shown in
what follows.

\[
\begin{align*}
\text{In[112]} &= \{ uA0 \times C = \\
& \quad \text{uA} / . \{ x \to \text{demand} \times C, y \to \text{demand} \times C, z \to z00 \} / . \\
& \quad \text{sol9} / . \text{wL} \to 1, \text{uA} \times E2 \} \\
\text{Out[112]} &= \{ 65.6617, 65.66174123232441454 \}
\end{align*}
\]

Thus, the conclusion for “the GE with public good on commodity tax” is exactly the
same as the one for “the GE with public good on income tax”.

\[
\begin{align*}
\text{In[113]} &= \text{TimeUsed[} ] \\
\text{Out[113]} &= 2.543
\end{align*}
\]

## IV : Poll Tax Provision of Public Good

In this section, we examine if it is possible to use poll tax instead to Lindahl tax in
order to achieve the optimum public good level, \( zO = 56.385555537178640578 \). Since
\( zO \) is provided for each member of the society, the utility function becomes the
following one \( u[y, x, zO] \).

\[
\begin{align*}
\text{In[114]} &= \text{uA0} = \text{uA} / . \text{z} \to \text{z00} \\
\text{Out[114]} &= \left(7.5090315978279543113 + \sqrt{x} + \sqrt{y}\right)^2
\end{align*}
\]

All the consumers maximize utility subject to income constraint:
max $u[y, x, zO]$ s.t. $p_y y + p_x x = (m_j - T/4)$ ($j = L, K, 1, 2$) \hspace{1cm} (19)

where $m_j$ is pre-tax income and $T$ is the tax to sustain $zO$ ($j = L, K, 1, 2, 3$). Worker (household $L$)’s pre-tax income, $m_L$, consists of initial endowment of labor, evaluated by the wage rate: $w_L L_{cA}$. It is assumed that they supply $L_{cA}$ for labor supply. Capitalist (household $K$)’s pre-tax income, $m_L$, consists of initial endowment of capital, evaluated by the rental price of capital: $w_K K_{cA}$. It is assumed that they supply $K_{cA}$ for capital supply. Entrepreneur 1 (household 1)’s pre-tax income, $m_1$, consists of profit for the sector 1, $\pi_{1A}$. Finally, entrepreneur 2 (household 2)’s pre-tax income, $m_2$, consists of profit for the sector 2, $\pi_{2A}$.

Demand function of workers for commodity $y$, $y_L^D$, that for commodity $x$, $x_L^D$, demand function of capitalists for commodity $y$, $y_K^D$, that for commodity $x$, $x_K^D$, demand function of entrepreneur 1 for commodity $y$, $y_{E1}^D$, that for commodity $x$, $x_{E1}^D$, and demand function of entrepreneur 2 for commodity $y$, $y_{E2}^D$, and that for commodity $x$, $x_{E2}^D$, are derived as in what follows.

In[115]:= \text{sol3D} = Solve[\{D[uA0, x] / D[uA0, y] = px / py, px * x + py * y = (m - T/4)\}, \{x, y\}][[1]]

Out[115]= \{x \to \frac{\text{py} \ (4 \ m - T)}{4 \ \text{px}} \ , \ y \to \frac{\text{px} \ (4 \ m - T)}{4 \ \text{py}} \}

In[116]:= demand$L$xD = x /. \text{sol3D} /. \{m \to wL \star LeA\};
demand$L$yD = y /. \text{sol3D} /. \{m \to wL \star LeA\};
demand$K$xD = x /. \text{sol3D} /. \{m \to wK \star KeA\};
demand$K$yD = y /. \text{sol3D} /. \{m \to wK \star KeA\};
demand$E1$xD = x /. \text{sol3D} /. \{m \to pi10\};
demand$E1$yD = y /. \text{sol3D} /. \{m \to pi10\};
demand$E2$xD = x /. \text{sol3D} /. \{m \to pi20\};
demand$E2$yD = y /. \text{sol3D} /. \{m \to pi20\};

In[117]:= demand$A$yD = Simplify[\text{demand$L$yD} + \text{demand$K$yD} + \text{demand$E1$yD} + \text{demand$E2$yD}];

In[118]:= demand$A$xD = Simplify[\text{demand$L$xD} + \text{demand$K$xD} + \text{demand$E1$xD} + \text{demand$E2$xD}];

General equilibrium for country A with public good when poll tax is adopted instead of Lindahl taxation, “GE with public good on commodity tax”, is defined by
General equilibrium for country A with public good when poll tax is adopted instead of Lindahl taxation, "GE with public good on commodity tax", is defined by

\[ y_A^D = y_A^S, \]  
\[ x_A^D = x_A^S, \]  
\[ p_z z = T, \]  
\[ L_{1A}^D + L_{2A}^D + L_{3A}^D = L_{eA}, \]  
\[ K_{1A}^D + K_{2A}^D + K_{3A}^D = K_{eA}. \]  

From the application of Newton method on (20) ~ (24) we compute "the GE with public good on poll tax" as in what follows.

\[
\text{In[119]:= check400 = }
\{\text{supply$A$x = demand$A$xD, supply$A$y = demand$A$yD, demand$K1 + demand$K2 + demand$K3 = KeA, demand$L1 + demand$L2 + demand$L3 = LeA, p_z * z00 = T} \} / . \{wL → 1, z → z00\};
\]

\[
\text{In[120]:= sol10 = FindRoot[check400, \{px, 1\}, \{py, 1\}, \{pz, 1\}, \{wK, 1\}, \{T, 1\}, AccuracyGoal → 30, WorkingPrecision → 20, MaxIterations → 1000]}
\]

\[
\text{Out[120]= \{px → 14.153545624140019458, py → 19.279639252690992305, pz → 4.610479019179988818, wK → 3.810370345862201692, T → 259.96442078998567379\}}
\]

Note that we have exactly the same equilibrium prices for commodities and inputs.

\[
\text{In[121]:= sol17}
\]

\[
\text{Out[121]= \{px → 14.153545624140019458, py → 19.279639252690992305, wK → 3.810370345862201692, t1 → 0.061932787239262383167, z → 56.38555537178640578, tL → 0.32500153599364843437, tK → 0.55737371673138213739\}}
\]

\[
\text{In[122]:= check400 /. sol10}
\]

\[
\text{Out[122]= \{True, True, True, True, True\}}
\]

Profit for the sector 3 is zero as shown in what follows.

\[
\text{In[123]:= (pz * z00 == wL * demand$L3 + wK * demand$K3) /. sol10 /.}
\quad\{wL → 1 /. z → z00}
\]

\[
\text{Out[123]= True}
\]
This solution, however, cannot be regarded as a genuine general equilibrium, since for this solution the incomes for entrepreneurs 1 and 2 are negative when this poll tax is imposed, as shown in what follows.

\[
\text{In[124]} := \{\text{income$A$E10D} = (\text{pi10} - T/4) / \text{wL} \rightarrow 1 / \text{sol10}, \\
\text{income$A$E10B}\}
\]

\[
\text{Out[124]} := \{-48.327516381250892247, 0.56326765368126488\}
\]

\[
\text{In[125]} := \{\text{income$A$E20D} = (\text{pi20} - T/4) / \text{wL} \rightarrow 1 / \text{sol10}, \\
\text{income$A$E20B}\}
\]

\[
\text{Out[125]} := \{-50.05770833356105865, 0.45546873059374796\}
\]

Thus, we cannot sustain $z_0$.

Conclusions

The aim of this paper is to integrate the two traditions of public economics. The first tradition asks, for instance, which taxation is desirable in order to impose a tax, income tax or commodity tax, without paying any attention on why the tax is necessary. The second one asserts that government imposes a tax in order to provide public goods. By constructing a primitive general equilibrium model which incorporates a public good, this paper asks which taxation is desirable in order to impose tax, income tax, commodity tax, or poll tax to provide the public good. Formally, in Section I, we utilized the Lindahl mechanism to compute a Pareto-optimal public good level. The burden-sharing in this Lindahl mechanism may be regarded as a tax on the society members, while utilizing pseudo-market mechanism. We called it Lindahl taxation. In Section II, we computed the rate of income tax in order to sustain the optimal public good level, and compared the Lindahl taxation and income tax. We obtained the conclusion, in which the income tax is more desirable than the Lindahl tax from the utilitarian viewpoint, as well as from the viewpoint of Gini coefficient. In Section III, we computed the rate of (proportional) commodity tax in order to sustain the optimal public good level, and compared the Lindahl tax, income tax and commodity tax. We obtained exactly the same conclusion as in Section II, since we have exactly the same general equilibrium except for the tax rates. Finally, in Section IV, we examined if it is possible to sustain the optimal public good level, showing the impossibility.

This paper adopted the simulation approach in which production and utility functions are specified, and the parameters on these functions are also specified. Thus, the conclusions may depend on the specification. Further simulation with random selection
of those parameters will be conducted in subsequent papers.

References


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